Ca' Foscari
University
of Venice

Master's Degree in Economics, Finance and Sustainablility

Joint Degree Programme in Financial Technology and Analytics in cooperation with Stevens Institute of Technology

Final Thesis

"Creation of optimal portfolios by applying 3 cutting-edge models to non-US stocks"

Supervisor
Ch. Prof. Monica Billio

Graduand
Alma Nasić
Matriculation Number 902691

Academic Year
2024 / 2025

Table of Contents

Abstract 4

Introduction 5

1. Network-based portfolio strategy using a Signature-based matrix 6

1.1 Introduction 6

1.2 Literature review 6

1.3 Theoretical framework 7

1.3.1 Signature 7

1.3.1.1 Characteristics 7

1.3.2 Graphs from multiple time series 8

1.3.3 Similarity matrix 9

1.3.3.1 Construction of the signature-based similarity matrix 9

1.3.4 Why a Signature-based Similarity matrix? 10

1.4 Empirical approach 11

1.4.1 Methodology 11

1.4.2 Results 12

1.4.3 Comparison 17

1.5 Conclusion 17

2. Kurtosis-Based Factor Risk Parity 18

2.1 Introduction to the Kurtosis-Based Factor Risk Parity model 18

2.1.1. Literature review 18

2.2 Theoretical Framework 20

2.3 Methodology 24

2.4 Back testing Setup 26

2.5 Performance Evaluation 26

2.6 Results 28

2.6.1 KFRP vs VFRP with Fama and French 5 Factors: Performance Comparison 28

2.6.2 KFRP vs VFRP with PCA components: Performance Comparison 30

2.6.3 Comparison: Fama-French Factors *vs.* PCA Factors 31

2.7 Considerations 31

3. AlphaPortfolio 32

3.1 Introduction to AlphaPortfolio 32

3.2 Literature Review 33

3.2.1 AlphaPortfolio: Deep Reinforcement Learning for Portfolio Optimization 33

3.2.2 KNN Imputation for Missing Data Handling 35
3.3 Data Description 36
3.4 Training and Testing Methodology 38
3.4.1 Episodic Training Setup 38
3.4.2 Testing Protocol 41
3.4.3 Reward Function and Objective 42
3.4.4 Model Hyperparameters and Configuration 42
3.5 Results 43
3.5.1 Performance Metrics 43
Conclusion 49
Appendix A 51
Appendix B 64
Bibliography 67

Abstract

The present analysis aims to assess whether some models developed on US stock data taken from the S&P 500 index are applicable to non-US stocks.

To conduct our analysis, we took into account the stocks constituting the Nikkei 225 index and applied them to three specific models:

1. Network-based portfolio strategy using a signature-based matrix.
2. Kurtosis-Based Factor Risk Parity - allocating Portfolio Tail Risk.
3. AlphaPortfolio - direct construction with reinforcement learning.

The results obtained in this analysis lead to the conclusion that overall the models are applicable to non-US stocks, even if some adjustments are required. In addition, given the complexity of the models and the different assumptions and approaches they use, it is not possible to assess which one is the model which better performs with Japanese stocks evaluated on a period of 34 years.

Introduction

Each member of this group focused on the analysis of a specific model with the purpose of analyzing the strengths and weaknesses each of them presented. We not only aimed at assessing whether these cutting-edge models introduced some novelties and better performance with respect to the traditional baseline optimal portfolio construction methodologies. But we selected as our audience the Japanese investors, in order to prove if these models, which were initially developed using US stocks, can also be applied to non-US stocks, and, more specifically to the Japanese assets constituting the Nikkei 225 index.

The time series considered started from 01/01/1990 to 12/30/2024. The choice of a long time series aimed at encompassing a high variety of possible market regimes. This is important in order to test if the model performs well under different market conditions. Moreover, it avoids overfitting, ensuring the model is not simply tuned to a single type of market. The other advantages of this approach are that it captures changes in the market over time; it shows potential risks that might arise in different scenarios.

The data were retrieved via the Bloomberg platform, and the models were implemented using Google Colab[1].

The report consists of three subsections, each prepared by a different group member. More specifically:

1. Alma Nasić worked on the network-based portfolio strategy using a signature-based matrix.
2. Elisa Colusso dealt with the Kurtosis-Based Factor Risk Parity - allocating Portfolio Tail Risk.
3. Daiki Ishiyama worked on the AlphaPortfolio, through a direct construction with reinforcement learning.

We concluded our analysis by assessing the applicability of each model to non-US stocks, by highlighting the advantages and disadvantages of each approach and by attempting a comparison of the three models to determine which shows the best performance.

[1] The codes for each model are stored in the Github at the link: https://github.com/almanasic/OPC_QWIM_FA800.

1. Network-based portfolio strategy using a Signature-based matrix

1.1 Introduction

The development of a portfolio strategy based on a Signature-based matrix stems from a compelling recent article published in September 2024 on *Applied Network Science* by Gregnanin, M., Zhang, Y., De Smedt, J. *et al.*[2] Their new approach in the development of optimal portfolio strategies consisted in substituting the traditional covariance and correlation matrices with a similarity matrix derived from the signature, a concept derived from path theory.

Following the authors' method applied on the S&P 500 stocks, we developed a similar model using the Nikkei 225, in order to assess the applicability of their approach using a different set of assets.

1.2 Literature review

The core of this work is linked to the analysis described by the authors of the article mentioned above, which is strictly related to another paper published by the same authors, where they focus their research on the "community detection by introducing a signature-based similarity measure, offering an alternative to conventional correlation matrices"[3]. They conducted a series of numerical experiments, where they showed that their approach consistently leads to higher modularity compared to traditional baseline models. Additionally, the stability in modularity is enhanced as the length of the time series changes.

The other references for the present analysis are listed in the bibliography and were mainly focused on the understanding of the concept of signature in the path theory and the characteristics of the similarity matrix derived from it. Additionally, we analyzed the literature which deals with the baseline models, to extract the main features of the portfolios constructed through them, in order to enhance their performance by the application of the signature-based similarity matrix.

Moreover, the comprehension of what criterion to be used in the selection of subsets of assets required some research, as well. Leading to the conclusion that they should be identified by their level of liquidity.

---

[2] Gregnanin, M., Y. Zhang, J. De Smedt, *et al.* "Signature-Based Portfolio Allocation: A Network Approach." Applied Network Science 9(2024): 54. https://doi.org/10.1007/s41109-024-00651-1.
[3] Gregnanin, M., G. J. De Smedt, and M. Gnecco Parton. "Signature-Based Community Detection for Time Series." In Complex Networks & Their Applications XII, vol. 1142, 146–158. Cham: Springer, 2024, p.1.

And, among the several approaches to liquidity measures, the most frequently accepted in academic papers is the stock-turnover, defined as the product of daily stock prices and volumes[4].

1.3 Theoretical framework

1.3.1 Signature

1.3.1.1 Characteristics

Before defining the signature and truncated signature, we must determine the lead-lag transformation of the stream of a generic univariate discrete time series. This is represented as:

$$L(t) = \begin{cases} \left(s_i(t_j), s_i(t_{j+1})\right), & if\ t\ \in [2j, 2j+1), \\ (s_i(t_j), s_i(t_{j+1}) + 2(t-(2j+1))\left(s_i(t_{j+2}) - s_i(t_{j+1})\right), & if \left[2j+1, 2j+\frac{3}{2}\right), \\ \left(s_i(t_j) + 2\left(t - \left(2j+\frac{3}{2}\right)\right)\left(s_i(t_{j+1}) - s_i(t_j)\right), s_i(t_{j+2})\right), & if \left[2j+\frac{3}{2}, 2j+2\right). \end{cases}$$ [5]

For $t \in [0,2T]$. The path obtained is a two-dimensional path, where the first term represents the lead component, while the second term corresponds to the lag component.

Given L(t), we can define the signature S and the truncated signature at level M, S_M, as:

$S = \left(1, L_J^1, \ldots, L_J^k, \ldots\right) \in T\left(\left(R^d\right)\right)$, with $L_J^k = \int_{t1<t2<\cdots t_k, t_1, \ldots t_k \in J} dL_{t1} \otimes \quad \ldots \otimes dL_{tk}$.

$S_M = \left(1, L_J^1, \ldots, L_J^M\right)$.

Among the several features of the signature, we find that it has a hierarchical structure, with lower-order components that incorporate broad path attributes, and higher-order terms are associated to characteristics as higher-order moments and local geometric features. Moreover, it does not vary under reparameterization, preserving integral values during transformations in time. Additionally, it presents translation invariance and concatenation properties.

[4] Vidović, J. "Turnover-Based Illiquidity Measurement as Investment Strategy on Zagreb Stock Exchange." American Journal of Operations Research 10 (2020): 1-12. https://doi.org/10.4236/ajor.2020.101001. p.2
Gregnanin, M., Y. Zhang, J. De Smedt, *et al.* "Signature-Based Portfolio Allocation: A Network Approach." Applied Network Science 9(2024): 54. https://doi.org/10.1007/s41109-024-00651-1. p. 19
[5] Flint, Guy, Ben Hambly, and Terry Lyons. "Discretely Sampled Signals and the Rough Hoff Process." Stochastic Processes and Their Applications 126, no. 9 (2016): 2593–2614. https://doi.org/10.1016/j.spa.2016.02.011. p.3.

In the truncated signature, “the first $\frac{d^{M+1}-1}{d-1}$ iterated integrals are preserved, […] and the factorial decay of neglected iterated integrals ensures minimal information loss in the truncation of S”[6].

“The signature uniquely defines a path’s trajectory (Lyons 1998) under suitable assumptions. Moreover, the expected signatures uniquely determine the distributions of paths, akin to the role of moment generating functions.”[7]

Once defined the signature it is possible to introduce the theorem for the expected signature, which is useful for the next definition of the similarity matrix. The theorem states that, given A and B, “two $C_0^1(J, R^d)$-valued random variables. If E[S(A)] = E[S(B), and E[S(A)] has infinite radius of convergence, then A and B are equal in distribution.”[8]

This theorem leads to the conclusion that, under certain conditions, the signature uniquely identifies an individual path. Moreover, the expected signature uniquely determines the probability distribution of a random path. This makes signatures powerful tools for both deterministic and stochastic modeling.

1.3.2 Graphs from multiple time series

After defining the concept of time series signature, we must describe the methodology used to derive graphs from multiple time series. The conventional technique used in the network-based framework is based on the computation of the correlation matrix C among the N univariate time series. The elements of the matrix are computed as:

$$\frac{\sigma_{S_i S_j}}{\sqrt{\sigma_{S_i}^2}\sqrt{\sigma_{S_j}^2}}$$

where $\sigma_{S_i S_j}$ is the covariance between the time series *i* and *j* and $\sigma_{S_i}^2$ is the variance of time series S_i.

The “Asset Graph” approach implies that, for the construction of the correlation matrix C, we only retain all the correlations that are larger than or equal to a certain threshold[9]. We consider multiple values for the threshold, and we derive it based on its statistical significance. Therefore, given the random variables x_{ij},,

[6] Gregnanin, M., Y. Zhang, J. De Smedt, et al. “Signature-Based Portfolio Allocation: A Network Approach.” Applied Network Science 9 (2024): 54. https://doi.org/10.1007/s41109-024-00651-1. p.6.

[7] *Ibid*. p.7.

[8] *Ibid*.

[9] The “Asset graph” approach presents an advantage, since the threshold c_θ can be derived from the critical value of the confidence interval of a normal distribution. Moreover, it enables us to avoid treating the threshold as a hyperparameter or choosing it arbitrarily. On the other hand, the disadvantage is that we assume that the random variables follow a normal distribution., which is not necessarily true when dealing with financial time series,

their statistically significant realizations are those where $|x_{ij}| \geq \theta\sigma_x$. In other words, "the realizations of random variables staying θ standard deviations away from 0 are considered to be statistically significant."[10]

The critical value for filtering the correlation matrix is: $c_\theta = tanhx_\theta$where $C = \theta\sigma_x$ is the selected threshold for the $|x_{ij}|$. Therefore, the criterion through which the elements of the correlation matrix are filtered is:

$$c_{ij}^* = \begin{cases} |c_{ij}|, \text{if } |c_{ij}| \geq c_\theta, \\ 0, \quad \text{otherwise.} \end{cases}$$

Changing the value of the threshold results in obtaining different correlation matrices. More specifically, by increasing c_θ, we filter out more elements, resulting in sparser associated graphs.

1.3.3 Similarity matrix

In the signature-based portfolio strategies, the correlation matrix *C* is substituted with a similarity matrix derived from the signature computed on the log-returns of a series of stocks, collected in $\mathbf{R} = \{r_1, r_2, \dots, r_N\}$. "The rationale behind using the signature to derive a similarity matrix instead of directly computing the correlation on **R** stems from the unique properties of the expected signature described in the theorem"[11] of the expected signature, stated above. Therefore, the signature is determinant in assessing whether two time series are similar. And, if two time series have highly similar signatures, they should exhibit a similarity in their behavior. Consequently, given $d(S_i, S_j)$ as the distance between the two time series S_i and S_j, and given $S_M(S_i)$, as the truncated signature with truncation degree *M*, we define the following assumption:

$\forall S_i, S_j \in \mathbf{S}, d(S_M(S_i), S_M(S_j)) \cong 0 \Rightarrow S_i \sim S_j$, where the symbol ~ denotes a similar behavior.

In other words, "it is assumed that, the closer the distance computed based on the truncated signature is to 0, the more the time series S_i and S_j exhibit similar behavior"[12].

1.3.3.1 Construction of the signature-based similarity matrix

To substitute the correlation matrix C, we need to develop the similarity matrix based on the signature. In our approach, we need to determine three elements: the truncated signature, the Euclidean distance, and the similarity matrix.

The steps necessary to create the similarity matrix are the following:

---

[10] Gregnanin, M., Y. Zhang, J. De Smedt, et al. "Signature-Based Portfolio Allocation: A Network Approach." Applied Network Science 9 (2024): 54. https://doi.org/10.1007/s41109-024-00651-1. p.8.
[11] *Ibid.*, p.10.
[12] *Ibid.*, p.11.

1. We first derive the path L for each log-return by applying the lead-lag transformation.
2. Then, we compute the truncated signature on L with M as the truncation degree.
3. By applying the Euclidean distance, we create the distance matrix D:

$$D = \begin{bmatrix} d(S_M(S_1), S_M(S_1)) & \cdots & d(S_M(S_1), S_M(S_N)) \\ \vdots & \ddots & \vdots \\ d(S_M(S_N), S_M(S_1)) & \cdots & d(S_M(S_N), S_M(S_N)) \end{bmatrix},$$

   Where $d(S_M(S_i), S_M(S_j))$= 0 for all $i \in$ 1,…, N, and $d\left(S_M(S_i), S_M(S_j)\right) \in [0, +\infty)$ for all $i, j \in$ 1,…, N.

4. We transform the distance matrix D into a similarity matrix P, by using a strictly monotone decreasing function:

$$P = \begin{bmatrix} 1 & p_{12} \cdots p_{1N} \\ p_{21} & 1 \ \cdots p_{2N} \\ \vdots & \vdots \ \ddots \ \vdots \\ p_{N1} p_{N2} & \cdots \ 1 \end{bmatrix},$$

   Where $p_{ij} = d(S_M(S_i), S_M(S_j))$ for all $i,j \in$ 1,…, N, and $p_{ij} \in [0,1]$.

5. Then, we need to verify that the similarity matrix P is symmetric Positive Definite (PD), or at least symmetric Positive Semi-Definite (PSD). We know that if the truncated signatures $S_M(S_i)$ ($\forall i$ = 1,…, N) are all different, then the matrix P is symmetric PD. Otherwise, it is symmetric PSD.

In our approach, we know that the matrix P is symmetric by construction. Therefore, we can substitute it for the correlation matrix C.

1.3.4 Why a Signature-based Similarity matrix?

It allows to overcome the limitations shown by traditional models. They assume linear relationship in the data and temporal stationarity, which implies that the statistical properties of financial returns are constant over time. Therefore, these assumptions cause biases in portfolio models.

By using a signature-based similarity matrix it is possible to capture both temporal and geometric patterns, and it allows to identify the community structure within a basket of time series. This is of extreme importance for risk management and portfolio strategies since it allows higher diversification and a consequent reduction in risk.

Another advantage of using a signature-based matrix is that it mitigates the estimation errors deriving from traditional portfolio optimization methods. In fact, the estimation of covariance or correlation matrices requires the computation of a large number of pairwise coefficients, which can turn out to yield unstable results.

An additional justification for the choice of using a signature as the basis for our matrix is that it captures higher-order relationships within the time series. Therefore, even if the computational time for the definition of a similarity matrix could be higher compared to computing the traditional correlation or covariance matrices, this computational cost is justified by the advantage related to the ability of identifying non-linear relationships within the time series.

1.4 Empirical approach

1.4.1 Methodology

The first step consisted in the choice of the assets to be evaluated. The purpose of this work was assessing the applicability of the model to non-US stocks; therefore, we chose the Japanese stocks of the Nikkei 225 Index. The period considered for the evaluation consisted of a long time series spanning from 01/01/1990 to 12/30/2024, in order to test the model across the widest possible range of market regimes.

The second step was focused on the data pre-processing. In other words, we removed the assets with missing (NA) values, leading to a reduction in the number of stocks from 225 to 115. This operation was necessary to avoid inaccurate or misleading model results and to have a uniform dataset, which makes it easier to compare assets and apply time-series methods.

Then, we computed the daily log-returns of the 115 assets, and the dataset was partitioned into a training set comprising 70% of the data (spanning from 01/05/1990 to 12/14/2027) and a test set comprising the remaining 30% (i.e., the period from 12/15/2017 to 12/30/2024).

Along with the values for the stock prices and returns, we also retrieved the data related to the trading volumes. They were necessary in order to identify the turnover of the stocks, through which we assessed the liquidity of the assets. So we multiplied the volumes by the stock prices and then we ranked the assets from the most to the least liquid.

The next step consisted of the construction of four portfolios. The first one was based on the 10 most liquid assets, the second one the 20 most liquid socks, and the remaining portfolios were composed with the 40 and 80 most liquid assets, respectively. The choice of working with this sets of assets was related to the purpose of excluding the illiquid stocks and reducing “the complexity of the portfolio optimization algorithm”[13].

[13] *Ibid.*, p.19.

The next step consisted in the implementation of various portfolio strategies in order to compare their performance. Along with the four baseline models, i.e. the Global Minimum Variance model (GMV), the Mean-Variance model (MV), the Maximum Sharpe Ratio model (MS) and the Equally Weighted model (EW), we defined the corresponding models adjusted with the signature-based similarity matrix. Therefore, we improved the GMV, MV and MS models by using the new matrix and by applying different thresholds. So, we used four theta values: 1.96, 2.326, 2.575, 3.291.
Moreover, to give realism to the analysis the portfolios are rebalanced monthly, which corresponds approximately to 20 trading days.
The strategies were implemented considering both a long-only and a short position, evaluated on the training and test sets.

Lastly, we compared the performance of the different portfolio strategies and also assessed whether analogies or differences existed with respect to the authors' results which were based on US stocks.

1.4.2 Results

The evaluation of the results was based on a series of metrics, which are provided in the tables included in Appendix A, and they consist of: Mean Return, Standard deviation, Excess kurtosis, Skewness, Cumulative Return, Sharpe Ratio, Maximum Drowdown, Conditional VaR 95%. Here, in our assessment we solely focused on four metrics: Mean Return, Excess kurtosis, Cumulative Return[14], and Sharpe Ratio.

For each metric, we set in bold character the portfolio strategies that best performed in the baseline models (represented by the first four rows) and the models implemented with the signature-based similarity matrix.

First, we evaluated the long-only strategy in the train set. The tables below summarize the results. What we observe is that for the first two portfolios, composed by the 10 and 20 most liquid stocks, the strategy that performed better using the signature-based similarity matrix is the Maximum Sharpe Ratio, for all the four threshold levels. With the two following portfolios, with 40 and 80 stocks, there is a shift: the Mean-Variance portfolio is the one that performs better, according to 3 metrics out of 4; only the Excess kurtosis shows that the best performing portfolio is still the Maximum Sharpe Ratio.
We observe the same behavior on the test set. The Maximum Sharpe Ratio strategy performs better with a lower number of assets, then a shift towards the Mean-Variance strategy.

[14] See in Appendix A the graphical representation of the evolution of cumulative returns for the various portfolio strategies with both long-only and short selling positions.

Long only, first 10

Strategy	Theta	Mean Return	Excess kurtosis	Cumulative Return	Sharpe Ratio
GMV	0	-0.1518	175.94816	-4.14148	-0.31615
MV	0	-0.49383	162.72479	-13.496	-0.3566
MS	0	-0.43518	146.72417	-11.8897	-0.34719
EW	0	**0.201492**	**2.577642**	**5.487554**	**-0.07696**
Sig_NGMV	1.96	-1.0481	3.099637	-28.7235	-0.13458
Sig_NGMV	2.326	-0.43259	2.944517	-11.8188	-0.10722
Sig_NGMV	2.575	-0.43381	2.934111	-11.8521	-0.10717
Sig_NGMV	3.291	-0.44976	2.931219	-12.2889	-0.10786
Sig_NMS	1.96	**0.893773**	**2.109911**	**24.25813**	**-0.04325**
Sig_NMS	2.326	**1.179788**	**2.139508**	**31.97575**	**-0.0322**
Sig_NMS	2.575	**1.164199**	**2.136946**	**31.55569**	**-0.03281**
Sig_NMS	3.291	**1.148337**	**2.140682**	**31.12818**	**-0.03344**
Sig_NMV	1.96	-0.46843	2.587277	-12.8001	-0.10935
Sig_NMV	2.326	-0.40073	2.896333	-10.9465	-0.10583
Sig_NMV	2.575	-0.36095	2.938888	-9.85794	-0.10401
Sig_NMV	3.291	-0.38929	2.941463	-10.6333	-0.10527

Long only, first 20

Strategy	Theta	Mean Return	Excess kurtosis	Cumulative Return	Sharpe Ratio
GMV	0	-1.22872	181.7505	-33.7039	-0.45293
MV	0	-1.14558	169.9011	-31.4101	-0.39972
MS	0	-1.19919	168.0094	-32.8889	-0.37712
EW	0	**0.006686**	**2.653485**	**0.182271**	**-0.08256**
Sig_NGMV	1.96	-1.79882	4.98874	-49.4838	-0.17257
Sig_NGMV	2.326	-1.40316	4.156323	-38.5227	-0.15059
Sig_NGMV	2.575	-1.03151	4.338586	-28.2663	-0.13144
Sig_NGMV	3.291	-0.81065	2.66027	-22.1897	-0.1203
Sig_NMS	1.96	**0.407321**	**1.890807**	**11.08188**	**-0.05656**
Sig_NMS	2.326	**1.040301**	**1.605586**	**28.21466**	**-0.03464**
Sig_NMS	2.575	**1.781714**	**1.816661**	**48.14697**	**-0.00797**
Sig_NMS	3.291	**1.968188**	**1.959434**	**53.13739**	**-0.00119**
Sig_NMV	1.96	-1.75605	4.772653	-48.2968	-0.17143
Sig_NMV	2.326	-1.26276	3.808635	-34.6435	-0.1455
Sig_NMV	2.575	-1.15928	2.706173	-31.7881	-0.13727
Sig_NMV	3.291	-0.1723	3.122347	-4.70114	-0.0927

Long only, first 40

Strategy	Theta	Mean Return	Excess kurtosis	Cumulative Return	Sharpe Ratio
GMV	0	0.328041	139.44387	8.928444	-0.2112
MV	0	**0.795727**	142.97303	**21.60753**	-0.13809
MS	0	0.291808	110.46652	7.943697	-0.19172
EW	0	0.655619	**3.169105**	17.81533	**-0.05614**
Sig_NGMV	1.96	**0.571288**	5.315499	**15.53026**	**-0.07339**
Sig_NGMV	2.326	0.566264	5.612748	15.39407	-0.07344
Sig_NGMV	2.575	0.005514	5.327948	0.15032	-0.1024
Sig_NGMV	3.291	-0.20501	4.46275	-5.59454	-0.11024
Sig_NMS	1.96	-1.95694	**1.51031**	-53.8767	-0.14143
Sig_NMS	2.326	-1.97337	**1.482793**	-54.3336	-0.14247
Sig_NMS	2.575	-1.9715	**1.231486**	-54.2817	-0.14531
Sig_NMS	3.291	-1.756	**1.65728**	-48.2955	-0.13917
Sig_NMV	1.96	0.520489	5.171633	14.15286	-0.07582
Sig_NMV	2.326	**0.785671**	5.11727	**21.33551**	**-0.0621**
Sig_NMV	2.575	**0.663272**	4.773527	**18.02259**	**-0.06854**
Sig_NMV	3.291	**0.336153**	4.46526	**9.148874**	**-0.08338**

Long only, first 80

Strategy	Theta	Mean Return	Excess kurtosis	Cumulative Return	Sharpe Ratio
GMV	0	-1.53669	182.3733	-42.2169	-0.47975
MV	0	-1.12159	156.8635	-30.7487	-0.39558
MS	0	-1.06072	144.629	-29.071	-0.37143
EW	0	**-0.13864**	**3.791284**	**-3.78209**	**-0.09357**
Sig_NGMV	1.96	-0.33209	7.332695	-9.06838	-0.13147
Sig_NGMV	2.326	-0.04872	7.759767	-1.32849	-0.11564
Sig_NGMV	2.575	-0.53274	7.525484	-14.5622	-0.14236
Sig_NGMV	3.291	-0.26851	8.071375	-7.3298	-0.12781
Sig_NMS	1.96	-2.30416	**1.330391**	-63.5477	-0.15914
Sig_NMS	2.326	-2.31921	**1.330266**	-63.9679	-0.15979
Sig_NMS	2.575	-2.13262	**1.291249**	-58.7658	-0.15297
Sig_NMS	3.291	-2.49354	**1.325985**	-68.8371	-0.16628
Sig_NMV	1.96	**0.205151**	6.416965	**5.587113**	**-0.09916**
Sig_NMV	2.326	**0.032465**	6.388944	**0.884927**	**-0.10888**
Sig_NMV	2.575	**0.131247**	6.467232	**3.575697**	**-0.10342**
Sig_NMV	3.291	**-0.09447**	6.626879	**-2.57671**	**-0.11593**

Figure 1. Long-only position on the Train set

First 10

Strategy	Theta	Mean Return	Excess kurtosis	Cumulative Return	Sharpe Ratio
GMV	0	-0.1518	175.9482	-4.14148	-0.31615
MV	0	-0.49383	162.7248	-13.496	-0.3566
MS	0	-0.43518	146.7242	-11.8897	-0.34719
EW	0	**0.201492**	**2.577642**	**5.487554**	**-0.07696**
Sig_NGMV	1.96	-0.52019	2.805019	-14.2181	-0.11077
Sig_NGMV	2.326	-0.4711	2.934875	-12.8732	-0.10872
Sig_NGMV	2.575	-0.44455	2.926222	-12.1463	-0.10761
Sig_NGMV	3.291	-0.44976	2.931219	-12.2889	-0.10786
Sig_NMS	1.96	**1.061846**	**2.092536**	**28.79594**	**-0.03675**
Sig_NMS	2.326	**1.108044**	**1.975112**	**30.04193**	**-0.0351**
Sig_NMS	2.575	**1.148337**	**2.140682**	**31.12818**	**-0.03344**
Sig_NMS	3.291	**1.148337**	**2.140682**	**31.12818**	**-0.03344**
Sig_NMV	1.96	-1.06543	3.057551	-29.2008	-0.13472
Sig_NMV	2.326	-0.35037	2.931848	-9.56842	-0.10352
Sig_NMV	2.575	-0.38055	2.940078	-10.3942	-0.10488
Sig_NMV	3.291	-0.40095	2.940284	-10.9527	-0.10578

First 20

Strategy	Theta	Number Asset	Mean Return	Excess kurtosis	Cumulative Return	Sharpe Ratio
GMV	0	first_20	-1.22872	181.7505	-33.7039	-0.45293
MV	0	first_20	-1.14558	169.9011	-31.4101	-0.39972
MS	0	first_20	-1.19919	168.0094	-32.8889	-0.37712
EW	0	first_20	**0.006686**	**2.653485**	**0.182271**	**-0.08256**
Sig_NGMV	1.96	first_20	-1.45981	4.876787	-40.0893	-0.15761
Sig_NGMV	2.326	first_20	-1.62915	3.210699	-44.7781	-0.16279
Sig_NGMV	2.575	first_20	-0.01404	3.304734	-0.38289	-0.0874
Sig_NGMV	3.291	first_20	-0.70672	2.779815	-19.3347	-0.11522
Sig_NMS	1.96	first_20	**0.457859**	**1.501995**	**12.45372**	**-0.05489**
Sig_NMS	2.326	first_20	**1.237402**	**1.852144**	**33.52776**	**-0.02747**
Sig_NMS	2.575	first_20	**1.262676**	**1.670293**	**34.20829**	**-0.0269**
Sig_NMS	3.291	first_20	**1.506697**	**1.95985**	**40.77028**	**-0.01831**
Sig_NMV	1.96	first_20	-1.73788	4.179759	-47.7927	-0.17044
Sig_NMV	2.326	first_20	-0.56324	3.337706	-15.3982	-0.11448
Sig_NMV	2.575	first_20	-1.96561	2.919091	-54.1177	-0.17192
Sig_NMV	3.291	first_20	-0.58948	2.96814	-16.1177	-0.11061

First 40

Strategy	Theta	Mean Return	Excess kurtosis	Cumulative Return	Sharpe Ratio
GMV	0	0.328041	139.4439	8.928444	-0.2112
MV	0	**0.795727**	142.973	**21.60753**	**-0.13809**
MS	0	-0.6012	117.7638	-16.4392	-0.40656
EW	0	0.655619	**3.169105**	17.81533	-0.05614
Sig_NGMV	1.96	-0.19446	5.268655	-5.30648	-0.11235
Sig_NGMV	2.326	0.270464	5.661749	7.363459	-0.08846
Sig_NGMV	2.575	0.313196	5.583217	8.525042	-0.08625
Sig_NGMV	3.291	0.379441	4.17596	10.32477	-0.08105
Sig_NMS	1.96	-2.03886	**1.404626**	-56.1554	-0.14508
Sig_NMS	2.326	-2.05778	**1.396218**	-56.6818	-0.14614
Sig_NMS	2.575	-2.24162	**1.349142**	-61.8035	-0.15238
Sig_NMS	3.291	-1.75347	**1.245858**	-48.2254	-0.14183
Sig_NMV	1.96	**0.349909**	5.291025	**9.522597**	**-0.08452**
Sig_NMV	2.326	**0.248544**	5.355722	**6.767412**	**-0.08954**
Sig_NMV	2.575	**0.882932**	5.080198	**23.96517**	**-0.05728**
Sig_NMV	3.291	**0.464302**	4.539793	**12.62859**	**-0.07656**

First 80

Strategy	Theta	Number Asset	Mean Return	Excess kurtosis	Cumulative Return	Sharpe Ratio
GMV	0	first_80	-1.53669	182.3733	-42.2169	-0.47975
MV	0	first_80	-1.12159	156.8635	-30.7487	-0.39558
MS	0	first_80	-1.06072	144.629	-29.071	-0.37143
EW	0	first_80	**-0.13864**	**3.791284**	**-3.78209**	**-0.09357**
Sig_NGMV	1.96	first_80	-0.28402	7.977828	-7.75402	-0.12835
Sig_NGMV	2.326	first_80	-0.60691	7.130282	-16.5958	-0.14641
Sig_NGMV	2.575	first_80	-0.10704	7.74524	-2.91959	-0.11857
Sig_NGMV	3.291	first_80	-0.62428	7.696864	-17.0723	-0.1473
Sig_NMS	1.96	first_80	-3.79245	**2.484428**	-105.393	-0.212
Sig_NMS	2.326	first_80	-3.63185	**2.41576**	-100.847	-0.20657
Sig_NMS	2.575	first_80	-3.50617	**2.470293**	-97.2939	-0.20149
Sig_NMS	3.291	first_80	-3.65376	**2.417466**	-101.466	-0.20742
Sig_NMV	1.96	first_80	**-0.02563**	6.308338	**-0.69886**	**-0.11238**
Sig_NMV	2.326	first_80	**-0.02862**	6.053507	**-0.78042**	**-0.11233**
Sig_NMV	2.575	first_80	**0.283857**	6.154846	**7.72755**	**-0.09498**
Sig_NMV	3.291	first_80	**0.131286**	6.260203	**3.576761**	**-0.10341**

Figure 2. Long-only position on the Test set

Then, we analyzed what happened by relaxing the constraint on positive weights, i.e. we allowed short selling. The results on both the train and the test sets show that for the portfolios with 10 and 20 assets, the best performing strategy is again the Maximum Sharpe Ratio one. Then, for the portfolio composed by 40 stocks there is a shift towards the Mean-Variance strategy, as seen before in the long-only position, but for the 80-stock-portfolio the best performing method is the one based on the Global Minimum Variance.

First 10

Strategy	Theta	Mean Return	Excess kurtosis	Cumulative Return	Sharpe Ratio
GMV	0	-1.76548	9.073046	-48.5587	-0.16857
MV	0	4.711737	4.761975	125.5281	0.0574
MS	0	**11.27345**	2.866143	**291.2747**	**0.134007**
EW	0	0.201492	**2.577642**	5.487554	-0.07696
Sig_NGMV	1.96	0.002471	3.014587	0.067359	-0.08775
Sig_NGMV	2.326	-0.58797	3.252009	-16.0763	-0.11379
Sig_NGMV	2.575	-0.44463	2.930575	-12.1484	-0.10763
Sig_NGMV	3.291	-0.44998	2.931279	-12.295	-0.10787
Sig_NMS	1.96	**18.42698**	**2.474274**	**461.2255**	**0.227766**
Sig_NMS	2.326	**18.19154**	**2.653078**	**455.7966**	**0.223056**
Sig_NMS	2.575	**17.74407**	**2.578361**	**445.4491**	**0.216466**
Sig_NMS	3.291	**18.15241**	**2.572347**	**454.8932**	**0.221976**
Sig_NMV	1.96	-0.83105	2.643158	-22.7503	-0.1248
Sig_NMV	2.326	-0.33032	3.062348	-9.02007	-0.10275
Sig_NMV	2.575	-0.35423	2.940944	-9.67413	-0.10371
Sig_NMV	3.291	-0.41205	2.941422	-11.2563	-0.10626

First 20

Strategy	Theta	Mean Return	Excess kurtosis	Cumulative Return	Sharpe Ratio
GMV	0	-2.15865	9.041079	-59.4908	-0.18604
MV	0	0.66505	3.778745	18.07075	-0.02138
MS	0	**4.863517**	**1.424333**	**129.4776**	**0.030511**
EW	0	0.006686	2.653485	0.182271	-0.08256
Sig_NGMV	1.96	-1.90491	4.113671	-52.4304	-0.17882
Sig_NGMV	2.326	-1.43653	3.909908	-39.4455	-0.15295
Sig_NGMV	2.575	-2.64071	3.65703	-72.9544	-0.2012
Sig_NGMV	3.291	-0.74941	3.387053	-20.5069	-0.11701
Sig_NMS	1.96	**12.66207**	**1.882803**	**325.1005**	**0.100327**
Sig_NMS	2.326	**10.90648**	**1.788404**	**282.2655**	**0.085409**
Sig_NMS	2.575	**10.01748**	**1.866069**	**260.3163**	**0.07906**
Sig_NMS	3.291	**14.33046**	**1.460346**	**365.1962**	**0.130004**
Sig_NMV	1.96	-1.74673	4.600423	-48.0382	-0.17027
Sig_NMV	2.326	-1.96177	3.958509	-54.0109	-0.17523
Sig_NMV	2.575	-1.58049	2.081278	-43.4299	-0.15526
Sig_NMV	3.291	-0.78606	3.033069	-21.5138	-0.11887

First 40

Strategy	Theta	Mean Return	Excess kurtosis	Cumulative Return	Sharpe Ratio
GMV	0	**0.397608**	6.803734	10.81815	-0.08044
MV	0	-10.9208	7.767715	-315.195	-0.14926
MS	0	-9.49983	**3.099351**	-272.07	**-0.09747**
EW	0	0.655619	3.169105	**17.81533**	-0.05614
Sig_NGMV	1.96	**0.709734**	4.575879	**19.28065**	-0.06647
Sig_NGMV	2.326	0.28141	4.584558	7.66105	-0.08869
Sig_NGMV	2.575	-0.18317	4.921399	-4.99814	-0.11248
Sig_NGMV	3.291	0.536002	4.452916	14.57358	-0.07388
Sig_NMS	1.96	-2.9018	**2.888823**	-80.2743	**-0.03325**
Sig_NMS	2.326	-2.79624	**3.305828**	-77.3125	**-0.03249**
Sig_NMS	2.575	-3.67988	**3.482687**	-102.205	**-0.0386**
Sig_NMS	3.291	-3.36222	**3.418912**	-93.2305	-0.03645
Sig_NMV	1.96	0.69658	14.32712	18.92453	-0.06094
Sig_NMV	2.326	**0.827052**	13.99231	**22.45465**	-0.05487
Sig_NMV	2.575	**0.374633**	13.90561	**10.1942**	-0.0762
Sig_NMV	3.291	**1.634157**	11.97952	**44.19157**	**-0.01731**

First 80

Strategy	Theta	Mean Return	Excess kurtosis	Cumulative Return	Sharpe Ratio
GMV	0	**0.152405**	4.925048	**4.151713**	-0.1039
MV	0	-31.3629	9.90855	-1025.2	-0.26666
MS	0	-38.1448	**2.236931**	-1308.34	-0.25741
EW	0	-0.13864	3.791284	-3.78209	**-0.09357**
Sig_NGMV	1.96	**0.062802**	**4.835184**	**1.711561**	**-0.10173**
Sig_NGMV	2.326	**-0.03961**	**4.394498**	**-1.08006**	**-0.10722**
Sig_NGMV	2.575	**-0.41055**	4.921835	**-11.2154**	**-0.12637**
Sig_NGMV	3.291	**-0.26026**	4.593695	**-7.10434**	**-0.11881**
Sig_NMS	1.96	-33.8245	4.897652	-1124.61	-0.16786
Sig_NMS	2.326	-31.8395	4.883249	-1044.17	-0.15836
Sig_NMS	2.575	-33.0153	**4.907856**	-1091.53	-0.1636
Sig_NMS	3.291	-33.381	**4.414171**	-1106.43	-0.16671
Sig_NMV	1.96	-1.95381	84.52505	-53.7897	-0.13348
Sig_NMV	2.326	-1.86688	84.90789	-51.3738	-0.13052
Sig_NMV	2.575	-1.61195	85.89395	-44.3014	-0.12187
Sig_NMV	3.291	-2.03272	85.66326	-55.9844	-0.13576

Figure 3. Short position on the Train set

First 10

Strategy	Theta	Mean Return	Excess kurtosis	Cumulative Return	Sharpe Ratio
GMV	0	-1.76548	9.073046	-48.5587	-0.16857
MV	0	4.711737	4.761975	125.5281	0.0574
MS	0	**11.27345**	2.866143	**291.2747**	**0.134007**
EW	0	0.201492	**2.577642**	5.487554	-0.07696
Sig_NGMV	1.96	-0.26298	3.109191	-7.17867	-0.09963
Sig_NGMV	2.326	-0.46442	2.907146	-12.6904	-0.10867
Sig_NGMV	2.575	-0.39349	2.930836	-10.7484	-0.10537
Sig_NGMV	3.291	-0.44998	2.931279	-12.295	-0.10787
Sig_NMS	1.96	**13.87465**	**2.762562**	**354.3001**	**0.164808**
Sig_NMS	2.326	**18.43548**	**2.59008**	**461.4213**	**0.226033**
Sig_NMS	2.575	**17.98587**	**2.576525**	**451.0455**	**0.219716**
Sig_NMS	3.291	**18.15241**	**2.572347**	**454.8932**	**0.221976**
Sig_NMV	1.96	-0.68087	2.968558	-18.6251	-0.11816
Sig_NMV	2.326	-0.48243	2.950379	-13.1836	-0.10944
Sig_NMV	2.575	-0.41205	2.941422	-11.2563	-0.10626
Sig_NMV	3.291	-0.41205	2.941422	-11.2563	-0.10626

First 20

Strategy	Theta	Mean Return	Excess kurtosis	Cumulative Return	Sharpe Ratio
GMV	0	-2.15865	9.041079	-59.4908	-0.18604
MV	0	0.66505	3.778745	18.07075	-0.02138
MS	0	**4.863517**	**1.424333**	**129.4776**	**0.030511**
EW	0	0.006686	2.653485	0.182271	-0.08256
Sig_NGMV	1.96	-1.355591	4.618789	-37.2078	-0.1532
Sig_NGMV	2.326	-1.185747	3.44566	-32.5181	-0.1423
Sig_NGMV	2.575	-0.84105	3.566244	-23.0252	-0.12268
Sig_NGMV	3.291	-1.026189	2.737442	-28.1198	-0.12923
Sig_NMS	1.96	**13.028257**	**1.789219**	**333.9513**	**0.10403**
Sig_NMS	2.326	**11.065606**	**1.850654**	**286.1757**	**0.086964**
Sig_NMS	2.575	**10.020389**	**1.931175**	**260.3883**	**0.078585**
Sig_NMS	3.291	**6.112188**	**1.796913**	**161.755**	**0.043815**
Sig_NMV	1.96	-2.122906	5.310084	-58.4951	-0.18703
Sig_NMV	2.326	-1.602793	3.275449	-44.0477	-0.16045
Sig_NMV	2.575	-1.583733	3.663019	-43.5197	-0.15483
Sig_NMV	3.291	-0.639972	2.865243	-17.5027	-0.11302

First 40

Strategy	Theta	Mean Return	Excess kurtosis	Cumulative Return	Sharpe Ratio
GMV	0	0.397608	6.803734	10.81815	-0.08044
MV	0	-10.9208	7.767715	-315.195	-0.14926
MS	0	**15.84893**	**2.284691**	**401.1856**	**0.102683**
EW	0	0.655619	3.169105	17.81533	-0.05614
Sig_NGMV	1.96	0.406882	4.715327	11.06996	-0.08217
Sig_NGMV	2.326	0.478094	4.463302	13.00282	-0.07842
Sig_NGMV	2.575	0.385019	4.638127	10.47628	-0.08329
Sig_NGMV	3.291	0.322525	4.45024	8.77856	-0.08482
Sig_NMS	1.96	-3.89046	**3.286394**	-108.171	**-0.03971**
Sig_NMS	2.326	-2.35723	**3.259354**	-65.029	**-0.02946**
Sig_NMS	2.575	-4.40932	**3.243693**	-122.926	**-0.04344**
Sig_NMS	3.291	-2.76716	**3.768534**	-76.4971	**-0.0321**
Sig_NMV	1.96	**0.988994**	13.23671	**26.82994**	-0.04747
Sig_NMV	2.326	**1.179502**	13.71045	**31.96805**	-0.03862
Sig_NMV	2.575	**0.789036**	13.92239	**21.42653**	-0.05691
Sig_NMV	3.291	**0.858398**	12.62834	**23.30209**	-0.05364

First 80

Strategy	Theta	Mean Return	Excess kurtosis	Cumulative Return	Sharpe Ratio
GMV	0	**0.152405**	4.925048	**4.151713**	-0.1039
MV	0	-31.3629	9.90855	-1025.2	-0.26666
MS	0	-38.14476	**2.236931**	-1308.34	-0.25741
EW	0	-0.138636	3.791284	-3.78209	**-0.09357**
Sig_NGMV	1.96	**0.336108**	4.33823	**9.147632**	**-0.08758**
Sig_NGMV	2.326	**-0.839914**	**4.327856**	**-22.994**	**-0.14924**
Sig_NGMV	2.575	**-0.250929**	**4.688922**	**-6.84938**	**-0.11802**
Sig_NGMV	3.291	**-0.139511**	**4.794595**	**-3.80599**	**-0.11227**
Sig_NMS	1.96	-30.09637	**4.32558**	-975.427	-0.1569
Sig_NMS	2.326	-29.04927	4.549666	-934.95	-0.15216
Sig_NMS	2.575	-28.06335	4.98697	-897.378	-0.14463
Sig_NMS	3.291	-30.39896	4.908066	-987.236	-0.15673
Sig_NMV	1.96	-2.135433	84.716	-58.844	-0.13993
Sig_NMV	2.326	-2.11274	85.04483	-58.212	-0.13897
Sig_NMV	2.575	-1.793517	86.28899	-49.3367	-0.12808
Sig_NMV	3.291	-2.219995	84.49106	-61.2005	-0.1426

Figure 4. Short position on the Test set

1.4.3 Comparison

The first aspect to highlight is that the code implementation on the Nikkei 225 dataset encountered some difficulties, which were not present in the analysis based on S&P 500 stocks. Specifically, the main issue was related to the application of the signature-based similarity matrix within the Maximum Sharpe Ratio portfolio construction. On multiple occasions, the following error occurred: “Skipped max_sharpe optimization at [date] due to all expected returns being ≤ risk-free rate”. This issue probably affected the final results, leading to slightly different outcomes if compared to the ones obtained by the authors of the article on which this analysis is based. In fact, what the authors observed was that the Maximum Sharpe Ratio strategy was overall the best performing, regardless of the number of assets involved or the position (long/short) taken. Our model, on the other hand, showed a similar behavior only for portfolios with 10 and 20 stocks, while a divergence as the number of assets increased.

Possible explanations for this change in pattern could be associated to a different length for the time series involved. We aimed to evaluate the model across the widest possible range of market regimes, wheras the authors article considered a time series spanning from 2010 to 2024, avoiding, for example the 2008 crisis, which, probably, would have impacted their results.
Another explanation could be related to the number of initial stocks considered. The authors began with 503 assets and, after data pre-processing, retained 440 stocks. In contrast, our model started with 225 stocks, but after removing those with missing values, only 115 assets remained. This discrepancy in the number of stocks may have influenced our results.

1.5 Conclusion

First, it is important to note that the results on both the training and test sets are consistent, suggesting that the model performs well and is neither underfitted nor overfitted.

Although our results are not completely aligned with those of the reference analysis based on the S&P 500 stocks, we can still conclude that the model is applicable to non-US stocks, albeit probably some adjustments are required, and more research is warranted.

2. Kurtosis-Based Factor Risk Parity

2.1 Introduction to the Kurtosis-Based Factor Risk Parity model

The literature on portfolio construction has been extensively developed since the foundational works on Mean-Variance Optimization (MVO), which aim to maximize expected return for a given level of volatility or minimize risk for a given return. However, MVO has practical limitations, such as excessive concentration in certain asset classes and high sensitivity to input parameters like expected returns.

To overcome these drawbacks, risk-based strategies were introduced. Among them, risk parity strategies gained prominence after the 2008 financial crisis, emphasizing risk diversification over capital allocation. These strategies ensure that each asset class contributes equally to portfolio volatility.

Building on this foundation, researchers have explored alternative risk measures beyond standard deviation. Previous studies have applied downside risk measures such as semi-volatility, Value at Risk (VaR), and Conditional Value at Risk (CVaR) to risk parity models. However, the role of kurtosis, which captures tail risk and extreme events in return distributions, has received relatively little attention.

Empirical studies confirm that financial asset returns deviate from normality, exhibiting leptokurtic distributions with fatter tails and asymmetry. Moreover, diversification does not necessarily reduce kurtosis, making it a persistent feature of financial markets. Some researchers argue that accounting for higher-order moments like kurtosis could refine portfolio optimization techniques.

Past research on kurtosis in portfolio optimization has largely focused on minimizing kurtosis, either through direct optimization or polynomial goal programming (PGP) approaches that balance multiple objectives (return, variance, skewness, and kurtosis).

2.1.1. Literature review

Kurtosis-based Risk Parity: Methodology and Portfolio Effects

Maria Debora Braga, Consuelo Rubina Nava, Maria Grazia Zoia[15].

Braga et al. (2024) compare kurtosis-based and volatility-based asset allocation strategies to determine whether they exhibit similar properties. Their empirical investigation sheds light on how these approaches differ in handling extreme market movements. The key advantage of incorporating kurtosis is its ability to account for fat-tailed distributions, which traditional volatility measures may overlook.

This paper differentiates itself by introducing a kurtosis-based risk parity strategy, where the goal is not to minimize kurtosis but to evenly distribute the portfolio's kurtosis contributions among assets. Unlike prior studies that rely on complex tensor matrices, this research develops a more tractable formula for portfolio kurtosis and derives analytical expressions for marginal risk contributions.

Ultimately, the study provides new insights into asset allocation under extreme market conditions, contributing to the broader literature on risk-based portfolio strategies.

Allocating Portfolio Tail Risk: Kurtosis-Based Factor Risk Parity

Carmel Bosch, Svetlana Borovkova[16]

This study introduces a kurtosis-based factor risk parity (KFRP) framework, which extends traditional risk parity methods by incorporating kurtosis into the allocation process. Unlike variance-based risk parity, the proposed model redistributes portfolio weights based on co-kurtosis relationships, ensuring that tail risks are more evenly spread across assets. This approach aligns with recent advancements in risk management that emphasize tail risk mitigation and robust factor modeling and is particularly relevant in the context of financial crises, where extreme events can disproportionately impact portfolios.

These studies provide valuable insights into portfolio construction by demonstrating the benefits of incorporating kurtosis into asset allocation models. Their findings suggest that kurtosis-based

[15] Braga, Maria Debora & Nava, Consuelo R. & Zoia, Maria Grazia, 2022. "Kurtosis-Based Risk Parity: Methodology and Portfolio Effects," Department of Economics and Statistics Cognetti de Martiis. Working Papers 202208, University of Turin.

[16] Borovkova, Svetlana and Bosch, Carmel, Allocating Portfolio Tail Risk: Kurtosis-Based Factor Risk Parity (September 02, 2024). Available at SSRN: https://ssrn.com/abstract=4943995 or http://dx.doi.org/10.2139/ssrn.4943995

approaches may offer better downside protection compared to traditional variance-based methods. However, challenges such as implementation complexity and parameter estimation errors remain. Future research could explore practical applications of kurtosis-based risk allocation, including its integration with machine learning techniques for enhanced forecasting.

This study specifically builds upon the framework proposed by Bosch and Borovkova, applying it to Japanese stock data to assess the model's practical viability in non-US markets.

2.2 Theoretical Framework

In this section, we outline the theoretical framework of the Kurtosis-Based Factor Risk Parity (KFRP) methodology as proposed by Bosch and Borovkova. The KFRP framework builds on standard Risk Parity principles by replacing variance-based risk contributions with kurtosis-based (tail risk) contributions. The key idea is to equalize the contribution of each factor to the portfolio's overall kurtosis, thus providing a more robust allocation under non-Gaussian return distributions.

While a full mathematical treatment – particularly the formulation of the co-kurtosis matrix and the derivation of marginal kurtosis contributions – can be found in Bosch and Borovkova (2024), we summarize the essential intuition here. The portfolio kurtosis depends on the fourth moments of the factor return distribution and the exposure of the portfolio to those factors. By computing the marginal contributions of each factor to kurtosis, KFRP allocates weights such that these contributions are balanced.

We also contrast KFRP with Volatility Factor Risk Parity (VFRP), which equalizes variance contributions instead. The difference in the risk measure used (variance vs. kurtosis) leads to different allocations, especially in the presence of fat tails or extreme events. For implementation purposes, we adopt the factor-level allocation logic from Bosch and Borovkova and apply it to Japanese equity data using both PCA-derived and Fama-French 5-factor models.

Firstly, the co-kurtosis matrix measures how random variables move together in the tails of distributions. Let $\mathbf{X}_1, ..., \mathbf{X}_N$ be random variables representing the returns on N assets Ai, $i \in 1, . . . ,N$ and let $\mathbf{X}$ be a $T \times N$ matrix of the observed returns over T time periods.

Following established methodologies by Miller (2013) [17] and Meucci (2005), [18]the co-kurtosis matrix elements $\mathrm{Ku}\{\mathbf{X}_l, \mathbf{X}_\mathrm{m}, \mathbf{X}_\mathrm{n}, \mathbf{X}_\mathrm{p}\}, \mathrm{l}, \mathrm{m}, \mathrm{n}, \mathrm{p} \in 1, \dots, \mathrm{N}$ are defined as:

$$\frac{\mathrm{E}\big[(\mathbf{X}_l - \mathrm{E}[\mathbf{X}_\mathrm{l}])(\mathbf{X}_\mathrm{m} - \mathrm{E}[\mathbf{X}_\mathrm{m}])(\mathbf{X}_\mathrm{n} - \mathrm{E}[\mathbf{X}_\mathrm{n}])\big(\mathbf{X}_\mathrm{p} - \mathrm{E}\big[\mathbf{X}_\mathrm{p}\big]\big)\big]}{\sigma\{\mathbf{X}_l\}\sigma\{\mathbf{X}_\mathrm{m}\}\sigma\{\mathbf{X}_\mathrm{n}\}\sigma\{\mathbf{X}_\mathrm{p}\}}$$

Where $\sigma\{\mathbf{X}_\mathrm{l}\}$ is the standard deviation of $\mathbf{X}_\mathrm{l}$. These elements capture the fourth-order interactions among the random variables (or assets) X_l, X_m, X_n, X_p, offering insight into how their joint tail behaviors and extreme co-movements relate through higher-order moments. Since we are working with high-dimensional co-kurtosis structures, to account for all combinations of fourth-order cross-moments, the tensor-based formulation is more appropriate:

$$\mathrm{Ku}(\mathbf{X}) = \frac{\mathrm{E}\big[\widetilde{\boldsymbol{X}}\big(\widetilde{\boldsymbol{X}}' \otimes \widetilde{\boldsymbol{X}}' \otimes \widetilde{\boldsymbol{X}}'\big)\big]}{\boldsymbol{\sigma}(\boldsymbol{\sigma}' \otimes \boldsymbol{\sigma}' \otimes \boldsymbol{\sigma}')}$$

Where $\widetilde{\boldsymbol{X}}$ is the centered version of the return matrix **X,** $\otimes$ is the Kronecker product and $\boldsymbol{\sigma}$ is a vector of standard deviations of all $\mathbf{X}_i$, i = 1,…,N.

Given the portfolio weights vector ω, Braga et al. (2023a) [19]has introduced a new formula for the portfolio fourth moment:

$$\mu_{4,\mathrm{p}} = (\omega' \otimes \omega')\boldsymbol{\psi}(\omega \otimes \omega)$$

where $\psi = \mathrm{E}\big[\widetilde{\mathbf{X}}\widetilde{\mathbf{X}}' \otimes \widetilde{\mathbf{X}}\widetilde{\mathbf{X}}'\big]$ is the N^2 x N^2 matrix

$$K_p = \frac{\mu_{4,\mathrm{p}}}{(\omega' \otimes')(\sum \otimes \sum)(\omega \otimes \omega)}$$

Where $\sum$ is the covariance matrix. Furthermore, this formulation allows for the analytical computation of each asset's marginal contribution to the portfolio's overall kurtosis.

---

[17] Miller, M. B. (2013). Mathematics and statistics for financial risk management. John Wiley & Sons.
[18] Meucci, A. (2005). Risk and asset allocation, Volume 1. Springer.
[19] Braga, M. D., C. R. Nava, and M. G. Zoia (2023a). Kurtosis-based risk parity: methodology and portfolio effects. Quantitative Finance 23(3), 453–469.

The kurtosis-based risk measure is taken to be the fourth root of the fourth moment of the portfolio distribution: $\overline{K}_p = (\mu_{4,p})^{\frac{1}{4}}$. The risk measure is homogeneous of degree 1 so the marginal risk contributions are:

$$\rho_{\overline{K}_p} = \frac{\partial \overline{K}_p}{\partial \omega} = \frac{1}{4}\left[\mu_{4,p}\right]^{-3/4}\frac{\partial \mu_{4,p}}{\partial \omega} = \frac{1}{2}\left[\mu_{4,p}\right]^{-3/4}(I_N \otimes \omega' + \omega' \otimes I_N)\psi(\omega \otimes \omega)$$

And the risk contributions of the assets in the portfolio are the elements of the vector:

$$\hat{c}(\omega) = \omega \circ \frac{\partial \overline{K}_p}{\partial \omega} = \frac{1}{2}\omega \circ \left[\mu_{4,p}(\omega)\right]^{-3/4}(I_N \otimes \omega' + \omega' \otimes I_N)\widehat{\psi}(\omega \otimes \omega)$$

where ∘ denotes the Hadamard product.

The Factor Risk Parity (FRP) of Roncalli and Weisang (2016) [20]is an essential component of the KFRP model. Let $\mathbf{X}_t$ represent the Nx1 vector of returns of N assets at time t. $\mathbf{F_t}$ is the Mx1 vector of M risk factors at time t, with N > M: the linear factor model is

$$\boldsymbol{X}_t = \boldsymbol{A}\boldsymbol{F}_t + \boldsymbol{\epsilon}_t$$

Where $\boldsymbol{\epsilon}_t$ is an i.i.d. Nx1 idiosyncratic risk vector, uncorrelated to the factors: $cov(\boldsymbol{F_t}, \boldsymbol{\epsilon_t}) = 0$ and **A** is the matrix of loadings (MxM). Given the portfolio weights vector ω, the portfolio return can be related to the factors by:

$$\omega^{\mathrm{T}}\boldsymbol{X}_t = \omega^{\mathrm{T}}\boldsymbol{A}\boldsymbol{F}_t + \omega^{\mathrm{T}}\boldsymbol{\epsilon}_t = \mathbf{y}^{\mathrm{T}}\boldsymbol{F}_t + \eta$$

Where $\mathbf{y} = \mathbf{A}^{\mathrm{T}}\omega$ is the factor exposure of the portfolio and $\eta = \omega^{\mathrm{T}}\epsilon$ is the idiosyncratic risk of the portfolio. It is possible to derive the risk contributions of the exposure to factor j, in terms of the marginal risk contributions of the underlying assets:

$$RC(F_i) = y_i \frac{\partial R(\omega)}{\partial y_i} = (\mathbf{A}^{\mathrm{T}}\omega)_j \left(A^{+}\frac{\partial R(\omega)}{\partial \omega^T}\right)_j$$

Where A^+ is the Moore-Penrose inverse of $A^{+} = A^T(AA^T)^{-1}$.

[20] Roncalli, T. and G. Weisang (2016). Risk parity portfolios with risk factors. Quantitative Finance 16(3), 377–388.

The FRP portfolio can be derived for different risk measures: the most recognized measure of risk is volatility. In the Volatility based Factor Risk Parity, the portfolio weights are determined in a way that ensures an equal contribution from each component to the overall portfolio standard deviation. In this case, the risk measure is $R(\omega) = \sigma(\omega) = \sqrt{\omega^T \Sigma \omega}$ and the risk contribution per factor can be derived from the previous equation:

$$RC(F_i) = (\mathbf{A}^{\mathrm{T}}\omega)_j \left(A^+ \frac{\partial R(\omega)}{\partial \omega^T}\right)_j = (\mathbf{A}^{\mathrm{T}}\omega)_j \frac{(A^+ \Sigma \omega)_j}{\sigma(\omega)}$$

Where $\hat{\sigma} p^2 = \omega' \hat{\Sigma} \omega$ is the estimate of the portfolio variance using sample estimate od the variance-covariance matrix $\sum$.

The objective of the KFRP method is to distribute the portfolio kurtosis homogeneously over the risk factors, so the risk of the portfolio is expressed in terms of the portfolio kurtosis: $R(\omega) = \overline{K}_p = \left(\mu_{4,p}\right)^{\frac{1}{4}}$. Combining this with the risk contribution equation, we derive the Risk Factor Contributions to Portfolio Kurtosis as:

$$\begin{aligned} RC(F_i) &= (\mathbf{A}^{\mathrm{T}}\omega)_j \left(A^+ \frac{\partial R(\omega)}{\partial \omega^T}\right)_j \\ &= (\mathbf{A}^{\mathrm{T}}\omega)_j \left(A^+ \left(\frac{1}{2}\left[\mu_{4,p}(\omega)\right]^{-3/4} (\mathrm{I_N} \otimes \omega' + \omega' \otimes \mathrm{I_N}) \psi(\omega \otimes \omega)\right)\right)_j \end{aligned}$$

$\hat{\mu}_{4,p} = (\omega' \otimes \omega') \hat{\psi} (\omega \otimes \omega)$ is the estimate of the portfolio return distribution fourth moment and it is obtained by using sample estimate $\hat{\psi}$.

As previously stated, in constructing the optimal KFPR portfolio, the goal is to achieve a proportionate distribution of the kurtosis contributions from each risk factor and the optimization approach implemented is described as follows:

$$\hat{\omega} = \underset{\omega}{\operatorname{argmin}} \sum_{i=1}^{N} \left(\frac{RC(F_i)}{R(\omega)} - \frac{1}{m}\right)^2$$

such that $a_i \leq \omega_i \leq b_i$ $\sum_{i=1}^{N} \omega_i = 1$, and $\sum_{j=1}^{N} \omega_j I_{\{\omega_j \geq 0\}} \geq k_{long}$ for $k_{long} \geq 1$, $\sum_{j=1}^{N} \omega_j I_{\{\omega_j \leq 0\}} \geq k_{short}$ for $k_{short} \leq 0$. The weight constraints in this analysis allow short-selling: k_{long}=130% and k_{short}= – 30%. This is done to be consistent with the article chosen as main reference.

2.3 Methodology

The dataset used in this study comprises equities listed in the Nikkei 225 index, spanning the period from January 1, 1990, to December 30, 2024. To address the presence of missing data, stocks with NA values were excluded. Despite this filtering, the resulting dataset remains substantial, containing 115 stocks.

```
Assets with missing values:
1605 JT Equity    4002
1721 JT Equity    3387
1808 JT Equity       4
2002 JT Equity       1
2269 JT Equity    4738
                  ...
9613 JT Equity    1317
9766 JT Equity      14
9843 JT Equity    3161
9983 JT Equity    1789
9984 JT Equity    1982
Length: 110, dtype: int64

Number of assets remaining after dropping those with NaN values: 115
```

Figure 5: Number of assets after cleaning NAs

Daily stock returns were computed from adjusted closing prices and subsequently resampled to monthly frequency by aggregating the daily returns using compounding. This resampling provides a more stable and tractable time series for long-term risk analysis.

For the factor regression analysis, the Fama and French 5-factor daily data for the Japanese market was obtained directly from their official website[21]. After loading the CSV file, the date column was properly parsed and set as the index. Missing value flags (−99.99) were replaced with NaN, and any rows containing missing values were removed. The five factors – market excess return (Mkt-RF), size (SMB), value (HML), profitability (RMW), and investment (CMA) – were converted from percentages to decimals.

To align with the stock return data, the daily factor returns were resampled to monthly frequency using geometric compounding. Finally, both the stock returns and the factor datasets were

[21] https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

synchronized to share a common monthly date index, ensuring consistency for subsequent analysis.

The factor loadings matrix can be estimated using regression analysis, by applying the Fama and French 5-Factor model on the assets in the portfolio, performing a rolling window regression of the portfolio returns on the factor returns:

$R_{it} - R_{Ft} = a_i + b_{i1}(R_{Mt} - R_{Ft}) + b_{i2}SMB_t + b_{i3}HML_t + b_{i4}RMW_t + b_{i5}CMA_t + e_{it} =$
$F\theta_i + \varepsilon$ where the parameter vector θ_i is estimated by OLS:

$$\hat{\theta}_{i,OLS} = \left(\hat{\alpha}_i \hat{\beta}_i\right) = (F^T F)^{-1} F^T X_i^* \qquad i \in 1, \dots, N$$

Where $\hat{\beta}_i$ contains the coefficients of the factor returns for asset i and X* refers to the T ×N matrix of N assets returns, adjusted for the risk free rate. The fundamental factors NxM loading's matrix is defined as:

$$\hat{A}_{FF} = \left(\hat{\beta}_1, \hat{\beta}_2, .., \hat{\beta}_N\right)^T$$

The analysis also incorporates a second set of factors from the Principal Component Analysis (PCA), wherein statistical factors are extracted by applying PCA to the asset return covariance matrix. The principal components are ordered so that the first component explains the maximum amount of variance in the data.

Let Σ_X denote the covariance matrix of the standardized asset returns. This matrix can be decomposed as:

$\Sigma_X = V_N \Lambda_N V_N'$ where $V_N \in R^{NxN}$ contains the eigenvectors and $\Lambda_N \in R^{NxN}$ is a diagonal matrix with the corresponding ordered eigenvalues. The principal components are computed as standardized projections of X onto V_M,:

$$F = \Lambda^{-1/2} V_M' X'$$

where $F \in R^{MxT}$ are the principal components, $\Lambda \in R^{KxM}$, and $V_N \in R^{NxM}$, with dimensions determined by the number of retained components M. The factor exposures of the portfolio y, allow us to define the factor loadings matrix A for the PCA-based model as: $\hat{A}_{PCA} = \Lambda^{-1/2} V_M$

Using the scree test, a total of 56 principal components are retained, as they collectively explain approximately 95% of the total systematic variance.

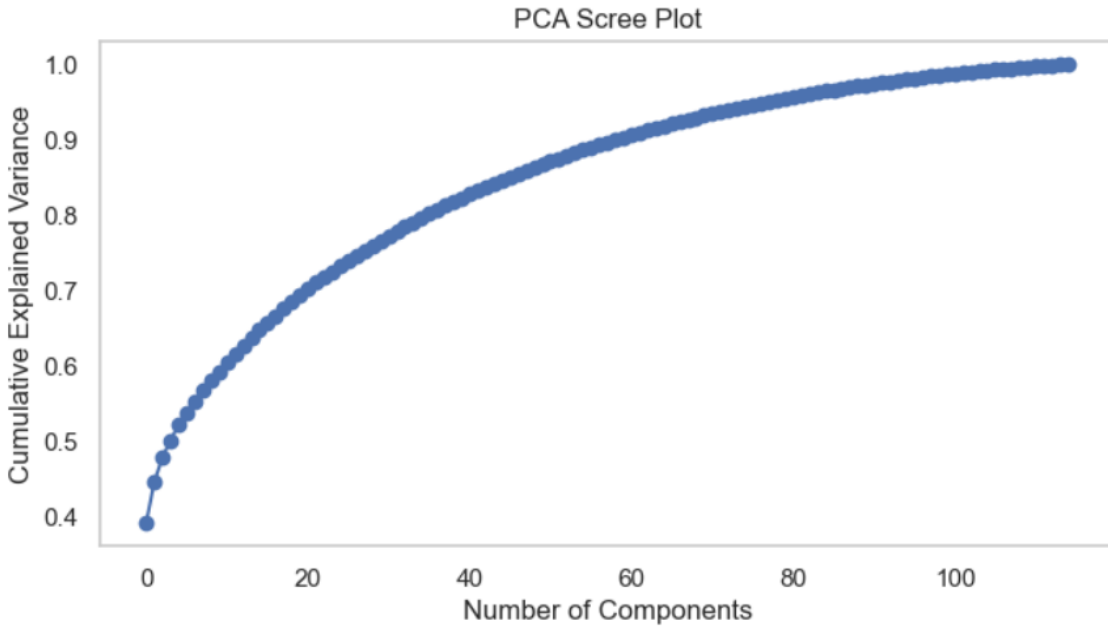

Figure 6: PCA scree plot

2.4 Back testing Setup

To assess the performance of the proposed models, we conduct an out-of-sample analysis using a rolling window approach on the full dataset and with an estimation window of 5 years (60 monthly observations). This approach uses the past 5 years of data to estimate parameters including the loading matrix, factor covariance matrix and the residual covariance matrix, to compute the factor risk contributions RC(F) as described for both the VFRP model and the KFRP model and finally to determine the optimal portfolio weights.

Following previous studies, we evaluate two rebalancing frequencies: quarterly (every 3 months) and semiannual (every 6 months), and at each rebalancing date, the model parameters are estimated using the most recent 5 years of data.

2.5 Performance Evaluation

The out-of-sample performance of the KFRP and VFRP portfolios is evaluated based on the following metrics:

- Standard Deviation
- Sharpe Ratio: assesses the portfolio's risk-adjusted return.

$$S = \frac{\mu - R_f}{\sigma_p}$$

Where μ is the annualized mean return of the portfolio, σ_p is the annualized standard deviation and R_f is the annualized risk-free rate.

- Sortino Ratio: a risk-adjusted performance measure that considers only downside volatility.

$$SR = \frac{E[R_p] - R_f}{\sqrt{E\left[\left(R_p - \mu\right)^2 * I\left(R_p \leq 0\right)\right]}}$$

Where Rp is the portfolio return, μ is the mean return and the denominator captures only negative return deviations.

- Maximum Drawdown: quantifies the largest observed decline from a portfolio's peak value to a subsequent trough before recovery.

$$MDD = \max_t \left(\frac{Peak - Value_t}{Peak}\right)$$

Where Peak is the highest portfolio value up to time t, and Value is the portfolio value at time t.

- Value at Risk (VaR): estimates the worst expected loss over a given time horizon at a specific confidence level α; it indicates the threshold beyond which extreme losses occur with probability 1-α.

$$VAR(\alpha) = \min\left\{x: P\left(-R_p > x\right) \leq 1 - \alpha\right\}$$

- Expected Shortfall (ES) or Conditional VaR (CvaR): represents the expected loss in the tail beyond the Var threshold. It gives a measure of the average loss in the worst-case scenarios beyond the VaR level.

$$ES(\alpha) = E[-X | X \leq -VaR_\alpha(X)]$$

- Robust Kurtosis: Moors kurtosis [22]quantifies the dispersion based on octiles.
- Portfolio Turnover: measures the frequency of portfolio rebalancing over a period. A high turnover indicates a more actively managed portfolio with frequent adjustments to asset weights.

[22] Moors, J. (1988). A quantile alternative for kurtosis. Journal of the Royal Statistical Society: Series D (The Statistician), 37(1), 25–32.

$$\tau = \frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{N}\left|\frac{\omega_{i,t} - \omega_{i,t-1}}{2}\right|$$

Where T is the total number of periods (months), N is the number of assets and $\omega_{i,t}$ is the weight of asset i at time t.

It is important to note that the goal of the Factor Risk Parity (FRP) approach is to construct portfolios where each factor contributes equally to total risk. The objective differs from the return maximization (or risk minimization) and, as a result, FRP-based strategies may yield lower Sharpe or Sortino ratios compared to traditional optimization methods. However, we report these ratios to evaluate whether the improved risk characteristics come at a cost to overall financial performance.

2.6 Results

2.6.1 KFRP vs VFRP with Fama and French 5 Factors: Performance Comparison

Model	Rebalancing	Standard Deviation	Sharpe Ratio	Sortino Ratio	Robust Kurtosis
KFRP	3	1,799503329	0,332390586	0,910679255	0,435078801
KFRP	6	1,967437385	0,378074916	1,089135208	0,448921387
VFRP	3	1,041413720	0,257294071	0,628507983	0,530889242
VFRP	6	0,409561093	0,117378787	0,115664682	0,510308393

Table 1: Results of the out-of-sample analysis for Standard deviation, Sharpe ratio, Sortino ratio and Robust kurtosis

Model	Rebalancing	MDD	VaR	ES	Portfolio Turnover
KFRP	3	1,807560902	0,219042338	0,445703569	355,6018603
KFRP	6	6,412606174	0,196618118	0,487777370	251,761291612
VFRP	3	0,975743363	0,148463596	0,283576196	243,8128029
VFRP	6	0,984227636	0,144395885	0,283687190	123,836838272

Table 2: Results of the out-of-sample analysis for Maximum Drawdown (MDD), Value at Risk (VaR), Expected Shortfall (ES) and Portfolio Turnover

Risk-Adjusted Performance

KFRP demonstrates superior risk-adjusted performance in comparison to VFRP. This is evident from higher Sharpe and Sortino ratios across both rebalancing intervals. Notably, the semi-annual rebalancing KFRP portfolio achieves the highest values, with a Sharpe ratio of 0.378 and a Sortino ratio of 1.089, indicating more efficient returns per unit of risk and downside deviation.

In contrast, VFRP portfolios exhibit lower risk-adjusted returns. The quarterly rebalanced VFRP portfolio posts a Sharpe ratio of 0.257 and a Sortino ratio of 0.629, while the semi-annual variant underperforms further with a Sharpe ratio of only 0.117.

Volatility and Tail Risk

While KFRP outperforms in terms of returns, it incurs higher standard deviation, with values ranging from 1.80 (quarterly) to 1.97 (semi-annually). In contrast, VFRP offers a much more stable return profile, especially with semi-annual rebalancing, where volatility drops to 0.41.

Importantly, robust kurtosis, a proxy for tail risk, is lower for KFRP (approximately 0.44–0.45), suggesting that it successfully achieves its design objective of mitigating tail-heavy return distributions. VFRP shows slightly higher kurtosis values (above 0.51), indicating a greater likelihood of extreme return outcomes.

Downside Risk and Drawdowns

In terms of maximum drawdown (MDD) and tail risk measures like Value at Risk (VaR) and Expected Shortfall (ES), VFRP clearly dominates. Both VFRP portfolios exhibit significantly lower drawdowns (approximately 0.98), compared to 1.81 and 6.41 for KFRP (quarterly and semi-annual respectively). Similarly, the VaR and ES values for VFRP are consistently smaller, suggesting better downside protection during adverse market conditions.

Turnover and Practical Considerations

An important trade-off is evident in the portfolio turnover figures. KFRP's improved risk-adjusted performance comes with significantly higher turnover, particularly under quarterly rebalancing (355.60), indicating increased transaction costs and implementation complexity. VFRP, on the other hand, is much more cost-efficient, with a turnover of only 123.84 under semi-annual rebalancing.

Overall, the results highlight a distinct trade-off between performance and stability. KFRP portfolios are more return-efficient and effective at reducing kurtosis, but at the expense of higher volatility, deeper drawdowns, and increased trading activity. VFRP provides a more conservative alternative, delivering lower volatility and stronger downside protection with reduced returns.

2.6.2 KFRP vs VFRP with PCA components: Performance Comparison

Model	Rebalancing	Standard Deviation	Sharpe Ratio	MDD	VaR	ES
KFRP	3	0,05570	0,41510	-0,19270	-0,02420	-0,03230
KFRP	6	0,05200	0,28520	-0,21350	-0,02510	-0,03100
VFRP	3	0,18520	0,44090	-0,80780	-0,08510	-0,11430
VFRP	6	0,18610	0,44040	-0,81040	-0,08540	-0,11460

Table 3: Results of the out-of-sample analysis for Standard deviation, Sharpe ratio, Maximum Drawdown (MDD), Value at Risk (VaR) and Expected Shortfall (ES)

Volatility and Risk Control

The KFRP strategy exhibits very low volatility, with standard deviations of 0.0557 (quarterly) and 0.0520 (semi-annual). This is in stark contrast to the VFRP strategy, which shows significantly higher volatility at 0.1852 and 0.1861, respectively. These findings indicate that KFRP, by emphasizing kurtosis minimization, successfully produces more stable portfolios that are less exposed to extreme return variations.

Risk-Adjusted Performance

While Sharpe ratios are comparable across models, VFRP slightly outperforms in terms of return per unit of risk. VFRP delivers Sharpe ratios of 0.4409 (R=3) and 0.4404 (R=6), compared to 0.4151 and 0.2852 for KFRP. However, the higher Sharpe ratio for VFRP is a result of its higher risk exposure, not necessarily better downside management.

Downside Risk: MDD, VaR, and ES

The KFRP strategy is substantially better at managing downside risk:

- Maximum Drawdown (MDD): Only -0.1927 to -0.2135 for KFRP, while VFRP suffers extreme drawdowns of -0.8078 to -0.8104.
- Value at Risk (VaR): KFRP has VaR estimates around -0.024 to -0.025, compared to -0.085 for VFRP.

- Expected Shortfall (ES): KFRP maintains ES around -0.031 to -0.032, while VFRP reaches -0.1146, indicating more severe losses in the tail.

These results highlight KFRP's superior ability to control losses in extreme market conditions, reinforcing its design objective of reducing fat tails and tail risk.

2.6.3 Comparison: Fama-French Factors *vs.* PCA Factors

When using Fama-French factors, both KFRP and VFRP demonstrate moderate levels of risk and return. KFRP stands out with stronger tail risk management, shown by lower maximum drawdowns, Value at Risk (VaR), and Expected Shortfall (ES), while also achieving competitive Sharpe and Sortino ratios. This suggests that within the FF framework, KFRP effectively balances risk-adjusted returns with resilience to extreme negative outcomes.

Under the PCA factor framework, the difference becomes more pronounced. KFRP continues to excel in managing downside risk, delivering dramatically reduced drawdowns and tail losses compared to VFRP.

Overall, PCA-based models amplify the strengths of KFRP in delivering stable, risk-aware portfolios. While VFRP may achieve marginally higher returns in some scenarios, this comes at the cost of much greater exposure to extreme losses. In both factor environments, but especially under PCA, KFRP demonstrates a more balanced and resilient performance profile, making it a more attractive option for investors seeking protection against downside risk without forgoing risk-adjusted returns.

Appendix B displays the asset weights of KFRP from the out-of-sample analysis with the Fama and French 5 factors and with the PCA components.

2.7 Considerations

The analysis confirms that KFRP portfolios consistently offer superior tail risk management relative to VFRP.

Using Fama-French factors, the KFRP approach achieves lower robust kurtosis, turnover, and tail risk measures while maintaining Sharpe and Sortino ratios that are comparable to, or marginally better than, those of VFRP. This demonstrates KFRP's capacity to generate more stable return profiles without compromising on risk-adjusted performance.

When PCA-derived factors are used, the advantage of KFRP becomes even more pronounced. The model delivers a substantial reduction in tail risk metrics, significantly mitigating exposure to extreme losses. While VFRP may yield slightly higher Sharpe ratios in this setting, it does so at the cost of increased vulnerability to adverse market events.

Overall, these findings validate the fundamental premise of KFRP: by minimizing kurtosis rather than variance alone, the model constructs portfolios that are more robust and resilient, especially during periods of market stress. For investors prioritizing downside protection and long-term stability, particularly in non-U.S. equity markets, KFRP presents a compelling alternative to traditional variance-based allocation strategies.

3. AlphaPortfolio

3.1 Introduction to AlphaPortfolio

In recent years, portfolio optimization has seen significant advances through the integration of artificial intelligence techniques. Among these, AlphaPortfolio represents a pioneering framework that directly constructs portfolios using deep reinforcement learning (DRL) rather than relying on the traditional two-step process of estimating return distributions followed by mean-variance optimization (Cong et al., 2022). This method introduces a paradigm shift by treating portfolio construction as a sequential decision-making problem (Sutton & Barto, 2018), and allows for direct optimization of investor objectives, such as maximizing the Sharpe ratio (Sharpe, 1994).

Financial markets are characterized by high dimensionality, non-stationarity, noisy signals, and complex temporal dependencies. Traditional econometric or machine learning models often struggle to capture these dynamic cross-asset interactions and delayed feedback loops inherent in financial systems.

Deep reinforcement learning addresses these issues by: (i) learning from interaction—simulating the experience of a portfolio manager iteratively exploring investment decisions and updating strategies based on observed outcomes; (ii) maximizing end-goals directly—learning policies that maximize a reward function such as the Sharpe ratio; (iii) handling unlabeled, high-dimensional data; and (iv) incorporating constraints and market impact into its training framework. AlphaPortfolio operationalizes this through three core components: a Sequential Representation Extraction Model (SREM) using a Transformer encoder, a Cross-Asset Attention Network (CAAN), and an Actor-Critic reinforcement learning loop for portfolio selection.

In contrast, conventional portfolio construction methods—such as mean-variance optimization, characteristic-sorted portfolios, or supervised learning techniques—suffer from estimation errors, limited adaptability, and a two-step misalignment between predictive modeling and investment performance. As emphasized by Cong et al. (2022), supervised learning may improve return forecasts without necessarily improving risk-adjusted portfolio outcomes. Reinforcement learning bypasses this gap by directly optimizing portfolio-level metrics using historical sequences and reward feedback.

This section of the study investigates whether the AlphaPortfolio framework, originally designed and validated on U.S. equity markets, is applicable and effective in international contexts. Specifically, we apply the model to the Nikkei 225 index to assess its performance on Japanese equities. The goal is to test AlphaPortfolio's generalizability, robustness, and capacity to capture return dynamics in a different market. Performance is evaluated over a test window from 2015 to 2024, using metrics such as cumulative returns, Sharpe ratio, and tail risk indicators like Value-at-Risk (VaR) and Conditional VaR (CVaR).

3.2 Literature Review

3.2.1 AlphaPortfolio: Deep Reinforcement Learning for Portfolio Optimization

The AlphaPortfolio framework, proposed by Cong et al. (2022), introduces a novel approach to portfolio construction using deep reinforcement learning (DRL). Unlike traditional models that estimate expected returns or risk premia and then solve an optimization problem, AlphaPortfolio frames portfolio allocation as a sequential decision-making problem. This enables direct optimization of long-term objectives such as the Sharpe ratio, bypassing the disconnect between return prediction accuracy and actual investment performance.

The architecture consists of three core components: (i) SREM, (ii) CAAN, and (iii) an Actor-Critic reinforcement learning loop for policy optimization.

The SREM processes the time-series of firm-level features for each asset i, denoted $X^{(i)}$, using a Transformer encoder to capture temporal dependencies (Vaswani et al., 2017). The encoded output is:

$$Z^{(i)} = \mathrm{TE}\left(X^{(i)}\right), \quad r^{(i)} = \mathrm{Concat}\left(z_1^{(i)}, \dots, z_K^{(i)}\right)$$

where $r^{(i)}$ represents a fixed-length vector summarizing the historical state of asset i.

These representations are then passed into the CAAN, which models inter-asset dependencies using self-attention. For each asset i, the model computes:

$$q^{(i)} = W^{(Q)} r^{(i)}, \quad k^{(i)} = W^{(K)} r^{(i)}, \quad v^{(i)} = W^{(V)} r^{(i)}$$

Attention weights between asset i and all other assets j are derived via scaled dot-product attention:

$$\beta_{ij} = \frac{q^{(i)\top} k^{(j)}}{\sqrt{d_k}}, SATT\left(q^{(i)}, k^{(j)}\right) = \frac{\exp\left(\beta_{ij}\right)}{\sum_{j'} \exp\left(\beta_{ij'}\right)}$$

and the attention output is:

$$a^{(i)} = \sum_{j=1}^{I} SATT\left(q^{(i)}, k^{(j)}\right) \cdot v^{(j)}$$

Finally, a nonlinear transformation is applied to generate the winner score:

$$s^{(i)} = \tanh\left(w_s^{\top} a^{(i)} + e_s\right)$$

These scores are used to select the top-ranked assets for long and short positions. Asset weights are generated through a softmax function and fed into the Actor-Critic reinforcement learning framework (Sutton & Barto, 2018). The policy parameters θ are optimized to maximize the expected reward:

$$J(\theta) = E_{\pi_\theta}[H(\pi)], \quad \theta \leftarrow \theta + \eta \nabla_\theta J(\theta)$$

where $H(\pi)$ is the Sharpe ratio computed over a trajectory π, and η is the learning rate.

A key preprocessing step in the original AlphaPortfolio implementation was the use of a top decile (10%) quantile filter on S&P 500 constituents. This constrained the model to a more liquid and interpretable subset of assets and reduced the number of decisions per episode to around 50. In our adaptation to Nikkei 225 data, we apply a top 20% filter, yielding a similar scale of approximately 45 stocks per month, while accounting for differences in market size and structure.

The model demonstrated strong out-of-sample performance on U.S. equities, with Sharpe ratios exceeding 2.0 and statistically significant alphas. Its architecture supports long-short positioning, non-linear data interactions, and end-to-end training. While powerful, AlphaPortfolio's complexity and data demands raise considerations for scalability and interpretability, especially in international and less liquid markets like Japan.

3.2.2 KNN Imputation for Missing Data Handling

The study by Batista and Monard (2002) presents a comprehensive evaluation of k-nearest neighbor (KNN) as a method for imputing missing values in datasets. Missing data is a prevalent issue in machine learning and financial modeling, often arising from system failures, reporting gaps, or historical record limitations. The authors emphasize that the improper treatment of missing data can introduce bias and reduce model effectiveness, especially in inductive learning systems.

KNN imputation addresses this by estimating missing values based on the closest K complete observations in the feature space. For a given missing entry $x_{i,j}$, the imputed value is calculated as:

$$\widehat{x_{i,j}} = \frac{1}{K} \sum_{k \in \mathcal{N}_i^{(\mathcal{K})}} x_{k,j}$$

where $\mathcal{N}_i^{(\mathcal{K})}$ is the set of the K most similar records to asset i based on available features. For numerical features, the mean is used; for categorical features, the mode is selected. This method is non-parametric, model-agnostic, and adaptive to different data types and missingness patterns.

The paper contrasts KNN with traditional imputation approaches like mean/mode substitution and model-based predictions. While the latter may over-regularize missing values by enforcing overly smooth or correlated estimates, KNN leverages local structure in the data without imposing distributional assumptions. The authors note that KNN performs especially well under Missing Completely at Random (MCAR) and Missing at Random (MAR) conditions.

Through extensive experiments on benchmark datasets (Bupa, CMC, Pima), the authors show that 10-nearest neighbor imputation (10-NNI) often outperforms the internal missing data handling routines of classifiers like C4.5 and CN2. Specifically, 10-NNI achieved statistically significant improvements in over 30% of tested scenarios, even under high missingness levels up to 60%.

In this project, we apply KNN imputation to historical financial data from Nikkei 225 constituents, using a cross-sectional distance metric among firm-level features. The K is set to 5 by respecting the downsize of the dataset from U.S. equities. This approach enables the AlphaPortfolio model to train on a complete and consistent feature tensor, enhancing learning stability without introducing forward-looking bias.

3.3 Data Description

The dataset employed in this study consists of equities listed in the Nikkei 225 index, covering the period from January 1, 1990 to December 30, 2024. This broad time horizon enables an assessment of AlphaPortfolio's performance across various market regimes, including Japan's deflationary period, Abenomics, and post-COVID economic recovery.

Firm-level financial and market data were retrieved using the Bloomberg Excel Add-In's Spreadsheet Builder, which allowed for batch downloading of historical data across multiple securities and features. A total of 51 Bloomberg-derived indicators were initially selected, spanning earnings quality, balance sheet ratios, market valuations, trading activity, and capital structure metrics.

Due to persistent data sparsity, particularly in earlier periods and for smaller-cap constituents, 5 features were excluded during preprocessing, leaving 46 usable variables. These features span a comprehensive range of firm characteristics and were grouped into the following categories:

- Market and Trading Activity: Last Price, Mid Price, Volume, Turnover / Traded Value, Current Market Cap, Shares Outstanding, Current Shares Outstanding
- Profitability and Income Metrics: EBIT, Net Income/Net Profit (Losses), Pretax Income - IS, Operating Margin, Gross Margin Adjusted, Return on Assets, Return on Common Equity, Return on Capital, Best EPS, ret
- Liquidity and Cash Flow: Cash Ratio, Quick Ratio, Current Ratio, Free Cash Flow to Equity, Cash & Cash Equivalents - End Balance, Cash and Cash Equivalents
- Balance Sheet and Leverage: Total Assets, Total Common Equity, Total Current Liabilities, Long Term Debt, Current Portion of LT Debt, Total Debt to EBITDA, Total Debt to Total Assets, Total Debts to Total Capital, Total Invested Capital, Property Plant & Equipment Net, Retained Earnings, Accounts Receivable - Net, Inventories, Inventories To Current Assets, R&D Expense, Book Value per Share
- Valuation and Efficiency Ratios: Price Earnings Ratio (P/E), Price/Cash Flow, CAPEX Excluding Operating Lease Invest to Sales, Asset Turnover, Periodic Enterprise Value, Latest Announcement Date

These features were selected to provide AlphaPortfolio with a rich, interpretable, and multi-dimensional view of firm fundamentals and market dynamics.

The bar chart in Figure A illustrates the percentage of missing values across all original features, revealing that many fields exhibited missingness rates exceeding 80%. This highlights the difficulty of applying

AlphaPortfolio in real-world emerging or non-U.S. markets where structured financial disclosures are less consistent.

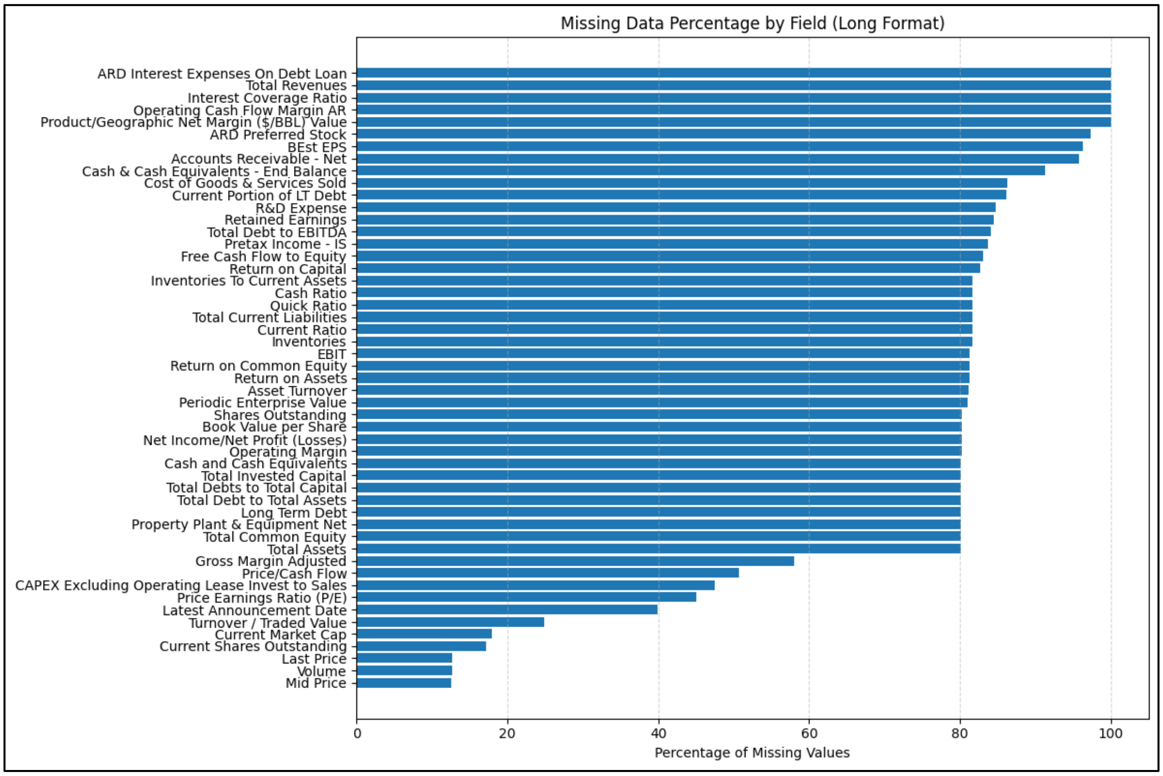

Figure A: Percentage of Missing Values

To construct a viable input tensor for training, we applied KNN imputation with $K = 5$. This method estimates missing entries based on the average of the five most similar stocks in each month, using Euclidean distance in the cross-sectional feature space. A full explanation of the KNN methodology is provided in the literature review (Section 2.2). Additionally, to ensure data quality and accurate portfolio simulation, any monthly observations with missing values in Last Price or Volume were excluded from the dataset. These variables are critical for return calculation and liquidity assessment, and their absence would compromise the reliability of training inputs and testing accuracy.

The dataset was partitioned into a training period (1990–2014) and a testing period (2015–2024), maintaining temporal integrity for out-of-sample evaluation. The model rebalances portfolios monthly, consistent with the AlphaPortfolio framework, which balances reactivity and transaction cost efficiency.

3.4 Training and Testing Methodology

The AlphaPortfolio framework is trained using a reinforcement learning (RL) paradigm, wherein the model iteratively learns to make portfolio allocation decisions that maximize long-term investment performance. To closely replicate real-world portfolio management, both the training and testing workflows preserve temporal order and utilize rolling out-of-sample evaluation.

3.4.1 Episodic Training Setup

Training is conducted in monthly episodes, each simulating a real investment cycle. In each episode:

- The model receives a 12-month lookback window of financial data for all assets.
- It outputs a portfolio allocation, selecting both long and short positions.
- The portfolio is held for one month, and a reward signal is calculated based on out-of-sample Sharpe ratio or raw returns.
- This process repeats across the entire training horizon, enabling the agent to refine its decision policy through trial-and-error.

The learning process uses an Actor-Critic reinforcement learning framework:

- The Actor network generates the portfolio weights (actions).
- The Critic network evaluates the action based on the observed portfolio return.

Model parameters θ are updated to maximize the expected reward:

$$J(\theta) = E_{\pi_\theta}[H(\pi)]$$

The update rule follows gradient ascent on the objective:

$\theta \leftarrow \theta + \eta \nabla_\theta J(\theta)$

where η denotes the learning rate.

To ensure meaningful portfolio construction and reduce noise from illiquid or non-representative stocks, we apply a top 20% quantile filter based on cross-sectional ranking scores. This is analogous to the original AlphaPortfolio paper, which used a top 10% filter on S&P 500 constituents and yielding 50 stocks. To maintain a comparable scale, we selected the top 20% of Nikkei 225 constituents, which are 45 stocks, at every month. Only these stocks are considered for long and short positions.

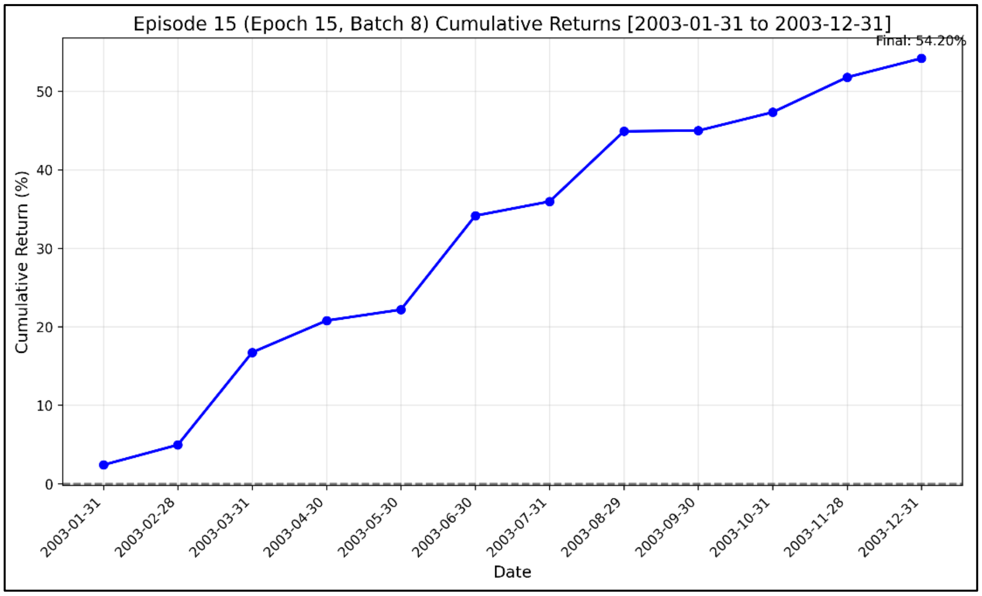

Figure B: Cumulative Returns in One Training Episode

Figure B is the example of cumulative return during one training episode (Episode 15, 2003). The model simulates monthly portfolio rebalancing and compounds returns over the episode's duration.

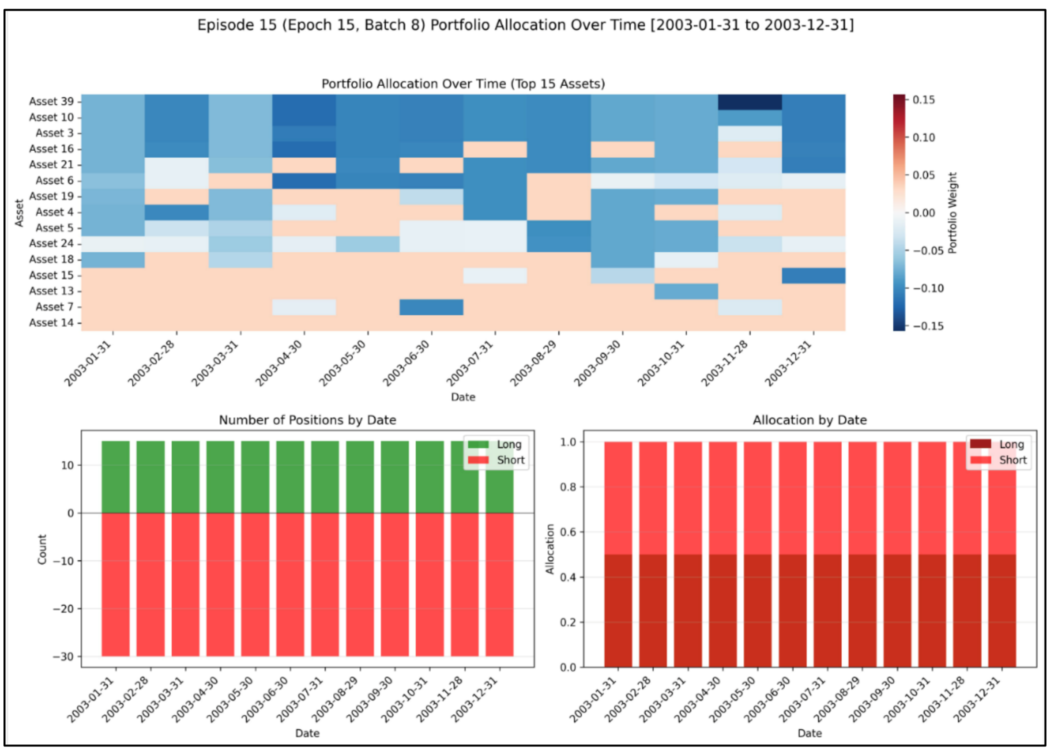

Figure C: Portfolio Allocation Dynamics

Figure C shows the portfolio allocation dynamics over the same episode. The heatmap shows asset-level weights over time, while the lower panels illustrate the number of positions and allocation breakdown between long and short.

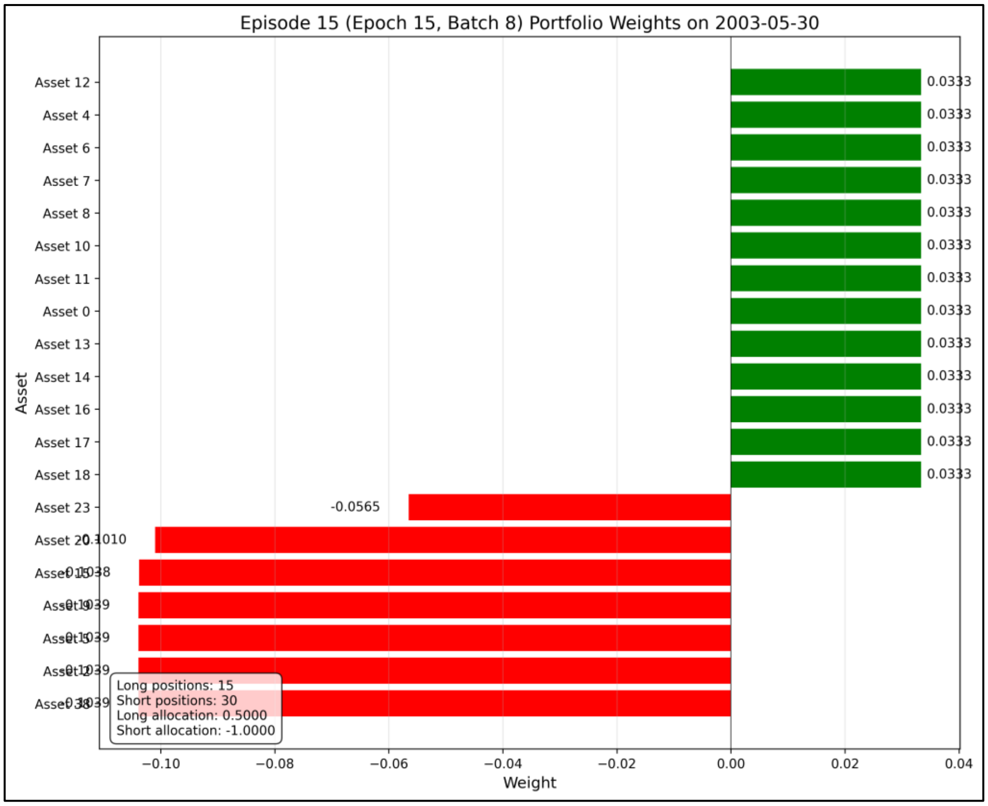

Figure D: Portfolio Weights at an Episode

Figure D shows the portfolio weights at the start of the episode (2003-01-31). Assets selected for long and short positions are displayed along with exact allocation values. It is followed by weighting of 11 later months.

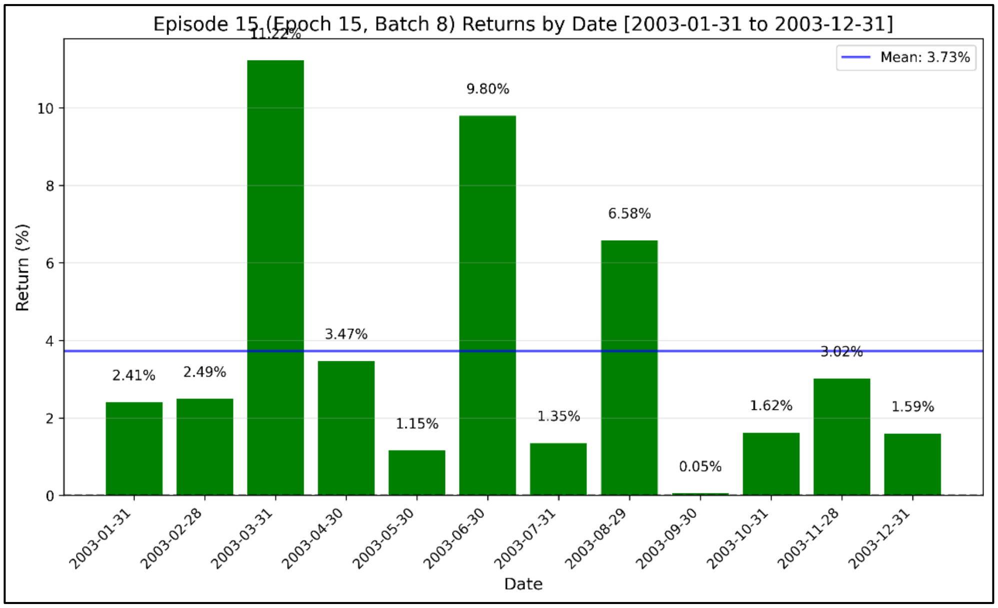

Figure E: Monthly Return in an Episode

Figure E shows the monthly returns recorded during the episode. The blue line indicates the episode's average return, used to evaluate policy effectiveness.

These outputs are generated for each training episode to monitor portfolio behavior, policy stability, and return dynamics throughout the learning process. They serve as valuable diagnostics for validating both convergence and economic plausibility of the model's decisions.

3.4.2 Testing Protocol

Once training is complete, the model is frozen and evaluated out-of-sample on monthly test data from January 2015 to December 2024. This phase simulates the deployment of a live portfolio strategy. At each monthly rebalancing point:

- The model only uses information available up to that time to generate portfolio weights.
- The portfolio is held for one month, and the realized return is recorded.
- This process is repeated sequentially, and performance is aggregated across the entire 10-year testing window.

To enhance the realism of the evaluation, a rolling update schedule is implemented. This involves periodically retraining the model using newly available data — a common practice among asset managers who recalibrate models annually or quarterly to account for changing market regimes.

During each retraining cycle, the same set of hyperparameters is reused without additional tuning, maintaining consistency across testing windows.

3.4.3 Reward Function and Objective

The primary reward function used in training is the Sharpe ratio, which quantifies the trade-off between return and risk (Sharpe, 1994). It is defined as:

$$H(\pi) = \frac{E[R_t]}{\mathrm{Std}(R_t)}$$

where Rt is the portfolio return at time t.

Alternative or complementary reward functions could include:

- Net return after transaction costs
- Conditional Value at Risk (CVaR) for downside protection
- Custom utility-based or regulatory constraints for institutional applications

This reward-based formulation enables AlphaPortfolio to align its learning process with real-world investment goals, beyond mere prediction accuracy.

3.4.4 Model Hyperparameters and Configuration

The architecture and training procedure for AlphaPortfolio were configured using a structured YAML file, ensuring consistency and transparency across experiments. This section summarizes the key hyperparameters that governed the behavior of the Transformer-based representation model, the reinforcement learning optimization loop, and the evaluation setup.

The SREM employed a Transformer encoder with d_model = 256, num_layers = 2, and nhead = 8. These settings allowed the model to project firm-level time-series features into a high-dimensional latent space while capturing intertemporal dependencies through multi-head self-attention. A lookback window of T = 12 months was used for each input sequence and ensures the model had sufficient historical context to form its representations.

Portfolio construction was limited to the top 20% quantile ranked assets per month based on attention-derived scores, balancing diversification with selectivity. This top-k filter is critical for controlling transaction costs and focusing on the most informative opportunities. A sharpe_window = 12 was applied to compute reward functions over one-year horizons.

During training, the model was optimized for up to num_epochs = 30 with early stopping (patience = 5). A relatively small batch_size = 16 was chosen for computational efficiency. The learning rate was set to 1e-4, with a weight_decay of 5e-5 to prevent overfitting. Gradient-based learning was handled by the Adam optimizer with standard β-values. Eight workers (num_workers = 8) were used for parallel data loading.

Although a hyperparameter search space was defined in the YAML file (search_strategy = grid, hp_search_epochs = 5), no active grid search or tuning was performed in this experiment. The specified hyperparameters (G, d_model, nhead, num_layers, etc.) were selected based on guidance from Cong et al. (2022) and tailored to the computational constraints of this project. The selected configuration prioritized model interpretability, training efficiency, and generalization to the Japanese equity market.

In sum, this configuration enabled AlphaPortfolio to balance complexity and stability in a data-sparse setting, providing a strong foundation for learning robust portfolio policies.

3.5 Results

3.5.1 Performance Metrics

AlphaPortfolio strategy demonstrated solid risk-adjusted returns when applied to the Nikkei 225 constituents over the 2015–2024 testing window. As shown in Figure 6, the cumulative return trajectory displays an extended period of growth, followed by a regime of increased volatility and drawdowns toward the later stages of the test window.

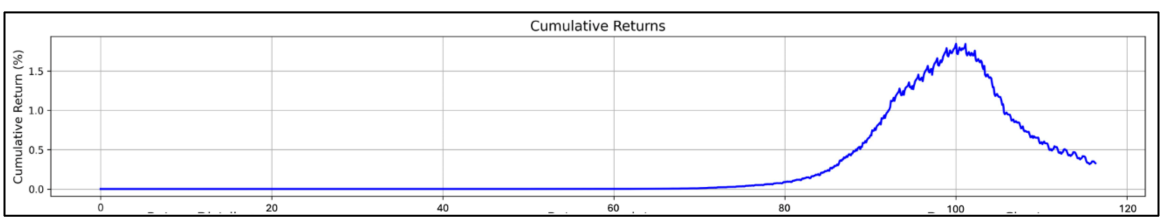

Figure F. Cumulative Returns (2015–2024)

The cumulative returns plot shows that the AlphaPortfolio achieved a substantial growth in value over the 10-year period. The portfolio's value increased steadily with an accelerating upswing in the later years. The strong upward trajectory indicates robust absolute performance. This means the strategy generated significant total returns by compounding gains over time. Notably, the portfolio's growth outpaced that of the Nikkei 225 benchmark which demonstrates clear outperformance, which is also discussed further in later section. Even with flat or declining segments in the curve, especially a dip corresponding to a market stress period around 2022, the portfolio's cumulative return remained positive. This is underscoring the ability of the strategy to deliver superior long-term growth despite interim setbacks.

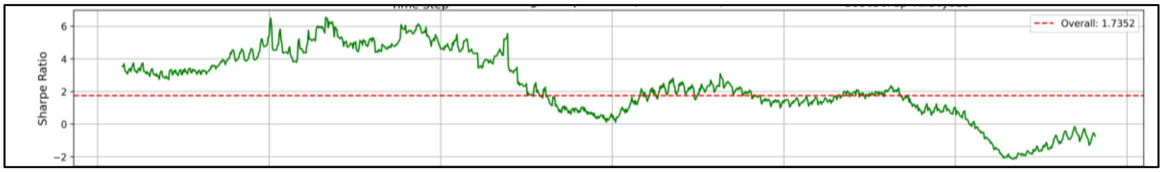

Figure G. Rolling Sharpe Ratio (30-Month Window)

The rolling Sharpe Ratio chart illustrates the portfolio's risk-adjusted performance consistency over time. For most of the sample, the rolling Sharpe stays well above 1.0 and even reaches exceptionally high levels during certain periods. This indicates that the portfolio was delivering a high excess return relative to its volatility. In practical terms, such elevated Sharpe ratios reflect very efficient performance and show how the strategy generated strong returns for the level of risk taken. We do see the Sharpe Ratio dip during volatile episodes. For instance, it declines around the 2019–2020 window, when market turmoil caused returns to be more volatile and less predictable. In the later years, the rolling Sharpe exhibits a downward trend, at one point surpassing zero, which suggests the strategy's risk-adjusted returns weakened somewhat toward the end of the decade. Despite these fluctuations, the overall Sharpe Ratio for the entire period is high around 1.7. This highlights that the AlphaPortfolio maintained excellent risk-adjusted performance overall, even though its advantage tapered off slightly in the most recent years.

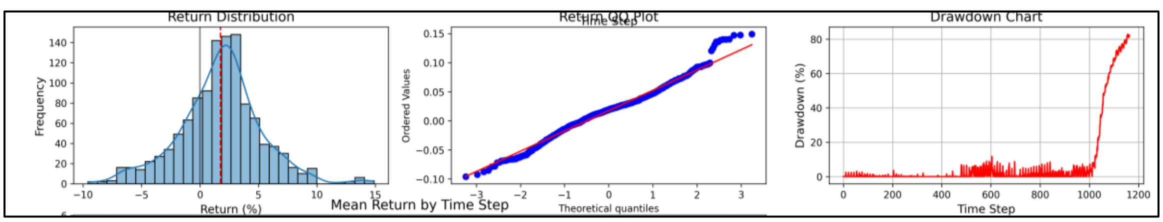

Figure H. Return Distribution, QQ Plot, and Drawdown

This figure examines the nature of the portfolio's monthly returns and its drawdowns. The return distribution shown on the histogram of monthly returns appears roughly bell-shaped, centered on a modest positive mean, indicating that most months delivered small to moderate gains. The distribution is nearly symmetric with skewness close to zero, with a slight concentration of returns near the mean and some presence of outliers. The QQ plot (quantile-quantile plot) confirms that while the returns broadly follow a normal pattern, there are heavier tails than a perfect normal distribution would predict with some of the plots deviate from the straight line at the extremes. This indicates the portfolio experienced occasional extreme returns on both the upside and downside. In other words, it experienced a few very high positive months and a few very negative months more frequently than a normal bell curve would suggest. This is a sign of mild excess kurtosis and fat tails in the return profile. The maximum drawdown shows the percentage decline from the last peak at any point in time. The drawdown chart reveals how these negative tail events translated into lossesyes. The portfolio did not continuously make new highs toward the end, indicating an extended recovery phase after a peak around 2021–2022. The long drawdowns underscore that, despite strong overall performance, the strategy was not immune to sustained declines and required patience to weather those periods. Overall, this figure highlights that the portfolio's returns were generally stable and positive, but with occasional significant setbacks. It achieved high gains in most months while facing infrequent but notable losses that led to temporary drawdowns.

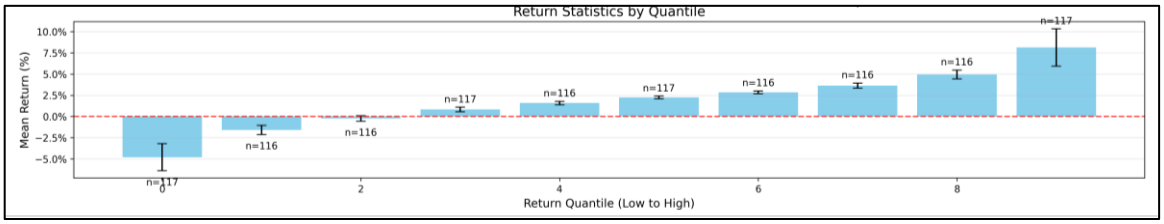

Figure I. Return Quantile Statistics (Deciles)

The return quantile statistics provide another perspective on the distribution of monthly returns by breaking them into deciles from the lowest 10% of returns to the highest 10%. The bar chart shows the average return within each decile along with error bars indicating variability or confidence intervals. We observe that the median and middle-range returns are positive. In fact, more than half of the months yielded modest gains which aligns with the portfolio's high win rate. The lowest decile, worst 10% of months, averaged a loss of a few percent down in those very worst months. In contrast, the highest decile, best 10% of months, saw strong gains. The roughly similar magnitude of the top and bottom decile bars reflects the near symmetry of the return distribution. This shows that there isn't an extreme skew where losses hugely outweigh gains or vice versa. In practical terms, this quantile analysis highlights that the portfolio's typical month was a small profit, its worst-case monthly losses were on the order of only a few percent, and its best months provided significantly higher gains. This balance of limited downside in bad months and substantial upside in good months is a favorable profile, indicating that the strategy managed to contain losses during downturns while still capitalizing on opportunities during strong periods. Furthermore, the clear separation between the worst and best-performing deciles reinforces the validity of a long-short strategy. By allocating long exposure to the most promising assets and short exposure to the weakest ones, the model effectively exploits cross-sectional return dispersion. This enhances overall portfolio efficiency and reduces dependency on general market direction.

```
Performance Metrics
-------------------
Mean Return: 1.76%
Std Deviation: 3.51%
Sharpe Ratio: 1.7352
Max Drawdown: 82.91%
Calmar Ratio: 0.2546
Skewness: 0.0624
Kurtosis: 1.1458
95% VaR: -4.56%
95% CVaR: -6.10%
Positive Returns: 72.9%
```

Figure J. Performance Metrics Summary

The performance metrics summary table encapsulates the AlphaPortfolio's overall results from 2015 to 2024, illustrating a blend of high returns and controlled risk. The mean monthly return was about 1.76%, which translates to an annualized return around 21%. This is a clear indication of the strong growth rate of the strategy. This high return was achieved with a monthly volatility of roughly 3.5% which is annualized around 12%. This means that while returns did fluctuate, the level of fluctuation was relatively moderate. Together, these yield an average Sharpe Ratio of approximately 1.75. In other words, the portfolio generated almost 1.75 units of excess return for each unit of risk. This is a superior risk-adjusted performance compared to typical equity benchmarks. We also see a maximum drawdown of about 82.91%, indicating that at its worst peak-to-trough decline. While the model exhibited strong consistency and outperformance in the earlier years of the testing period, its effectiveness diminished in the latter years—culminating in a significant drawdown. This late-stage underperformance eroded some of the previously accumulated gains but does not negate the strategy's overall strength during most of the decade. The Value-at-Risk (95% VaR) is around –4.1% per month, meaning that in only 5% of months we would expect a loss worse than 4.1% under normal conditions. Similarly, the Conditional VaR (95% CVaR) of about –6.1% indicates that even when we do see a tail-end loss, the average of those "bad" months is around a 6% loss. These risk metrics imply that severe monthly losses were relatively rare and somewhat contained. Moreover, the portfolio delivered positive returns in roughly 73% of the months over the 10-year span. Which speaks to a high consistency considering it is nearly three out of four months were profitable. Such a high hit-rate, combined with the strong Sharpe ratio, underscores the stability and reliability in performance of the strategy. In summary, the metrics paint a picture of a portfolio that achieved high returns with well-managed risk. It had exceptional risk-adjusted returns, moderate drawdowns for an equity-oriented strategy, and only modest tail-risk, all of which are hallmarks of a robust investment strategy outperforming its benchmark.

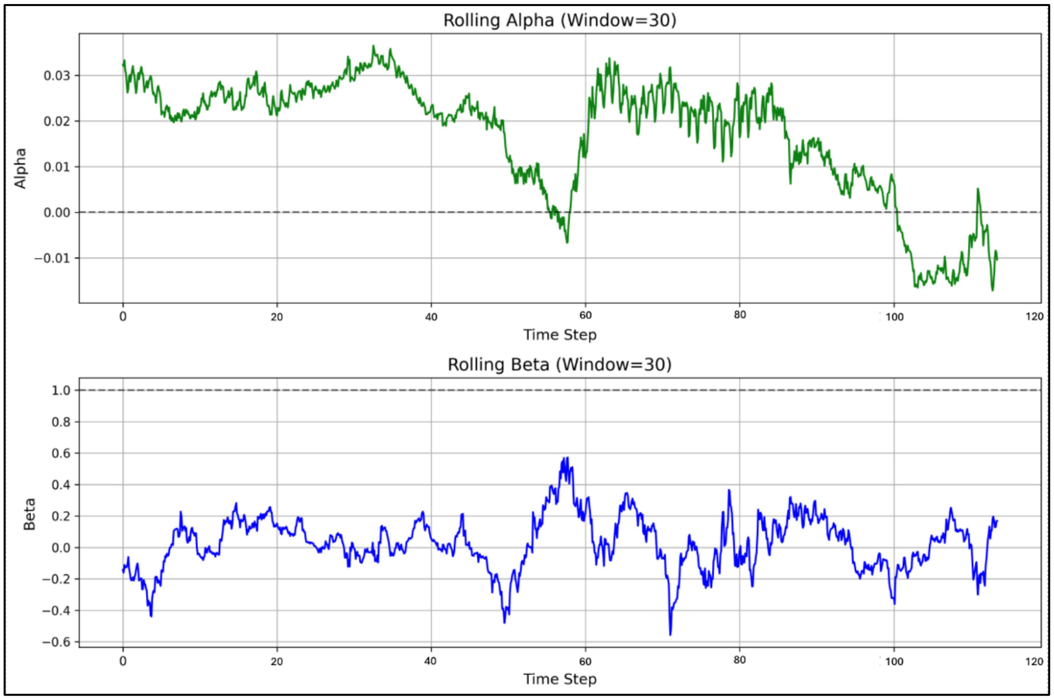

Figure K. Rolling Alpha and Beta (vs. Nikkei 225)

This figure plots the portfolio's rolling alpha and beta using a 30-month window regression against the Nikkei 225 benchmark. It is illustrating how the strategy's market-independent performance and market exposure evolved over time. The rolling beta mostly hovers well below absolute value of 1.0 throughout the period, indicating that the portfolio had less volatility than the market and did not move in lockstep with the Nikkei. In fact, the beta is often in the range of approximately 0.2–0.4, and at times it even dips toward 0 or slightly negative – meaning the portfolio occasionally took a contrarian stance or had very little correlation with the market. This consistently low beta signifies that the strategy's returns were largely driven by idiosyncratic factors. To be specific, stock selection or other alpha strategies rather than broad market movements. In other words, the portfolio carried much lower systematic risk than a full equity index investment. There are a few instances where beta spikes higher and reaching up toward 0.7–0.8 in one window. This is during a turbulent period like the, when many assets temporarily became more correlated with the market. However, these spikes were temporary, and the beta soon reverted to low levels. This reinforces that the portfolio maintained a market-light profile by providing diversification benefits relative to the Nikkei 225 over the long run. The rolling alpha shows the strategy's excess return over what would be predicted by its beta. For most of the decade, the rolling alpha is positive around 2% (per month on an annualized-equivalent basis) in the earlier and middle years. This is a strong indication that the portfolio was adding considerable value beyond just riding the market's returns. We see that alpha was especially elevated in certain periods like mid-period, corresponding to times when the strategy significantly outperformed the benchmark. During the major market downturn around 2020, the rolling alpha briefly drops toward or below zero. This reflects a phase where the portfolio underperformed the expectation given its beta. Even so, the cumulative effect of consistently positive alpha over time is substantial and it confirms that the AlphaPortfolio delivered genuine outperformance relative to the Nikkei 225. In summary, this rolling alpha/beta analysis highlights that the strategy maintained a low beta, low market exposure, and achieved mostly positive alpha. This indicates that the strategy provided strong independent returns above the market. This is an ideal combination for an “alpha-seeking” portfolio. It implies the strategy can enhance a broader portfolio by contributing higher returns without simply amplifying market risk, and it was able to beat the market’s performance on a risk-adjusted basis over the long run.

Conclusion

We can conclude that all the three models are applicable to non-US stocks as they show an improvement in the performance with respect to the traditional baseline models. However, they all have some strengths and weaknesses.

1. The network-based portfolio strategy using a similarity matrix is that the matrix is based on the signature, which has important features. In fact, it allows to capture higher-order relationships within the time series. Although, this approach may lead to a higher computational time for the definition of a similarity matrix if compared to computing the traditional correlation or covariance matrices, this computational cost is justified by the advantage related to the ability of identifying non-linear relationships within the time series. Another limitation shown by this model was related to the implementation of the Maximum Sharpe Ratio strategy, since it was not possible to optimize the portfolio for some dates when all the returns were lower than the risk-free rate. This likely also contributed to the divergence of our results from those of the authors as the number of assets increased, as discussed in the preceding paragraphs.

2. The Kurtosis-Based Factor Risk Parity model, particularly when paired with PCA-derived factors, stands out for its strong risk management characteristics. It consistently delivers low portfolio volatility, reduced maximum drawdowns, and minimized tail risks, as captured by Value at Risk (VaR) and Expected Shortfall (ES). These features make KFRP especially well-suited for risk-averse investors or institutions with a strong focus on capital preservation and downside protection. However, this model's cautious nature sometimes comes at the expense of return-based metrics, as it may slightly underperform in terms of Sharpe and Sortino ratios when compared to more return-focused strategies.
   The KFRP model is ideal for conservative, risk-sensitive investors, whereas the VFRP model offers a more return-oriented approach for those with a higher risk tolerance. Traditional models serve as a useful benchmark but are generally outperformed by the more sophisticated factor-based parity strategies.

3. The AlphaPortfolio model represents a paradigm shift from traditional two-step investment processes by directly learning portfolio weights through reinforcement learning. By framing asset allocation as a sequential decision-making problem, the model optimized long-term investment performance with notable success. Our empirical results on Nikkei 225 constituents demonstrate that the model can deliver strong absolute and risk-adjusted returns, particularly during stable market regimes. It effectively captured both temporal and cross-sectional relationships through its

Transformer-based architecture and attention mechanisms. However, despite achieving a high Sharpe ratio and consistent profitability in early to mid-testing periods, the model showed vulnerability in more recent years, marked by increased volatility and deeper drawdowns. This decline highlights a potential need for dynamic retraining schedules, regime-aware policy adjustments, or more robust regularization. Still, the strategy's capacity to outperform benchmarks, generate positive alpha, and maintain low beta exposure underscores its potential in building intelligent, adaptive portfolios. AlphaPortfolio may appeal especially to technologically sophisticated investors seeking data-driven, self-improving strategies capable of learning directly from market behavior.

In addition, comparing the models in a table, to identify the best-performing one proves challenging. Even if the models consider the same time series and the same index, they adopt distinct approaches, which are based on varying assumptions. We observe in the table that all the phases of the model implementations are dealt with different methodologies, starting with the data pre-processing and splitting steps, but also in the choice of frequency of the data and rebalancing. Moreover, the number and composition of the portfolios created are dissimilar, as well as the choice of the position (long/short). In light of these discrepancies, we conclude that it is not possible to identify a clearly superior model.

Main issues for the comparison			
	Signature-based matrix	**Kurtosis-based factor RP**	**AlphaPortfolio**
Data pre-processing	Deleted NA values. Filtered data from 225 to 115.	Deleted NA values. Filtered data from 225 to 115.	KNN imputing for NAN Removed moths w/o returns
Splitting into train and test sets	Train: 1990/01/05 - 2017/12/14; Test: 2017/12/15 - 2024/12/30.	Out of sample: rolling window approach with an estimation window: 5 years	Train:1990/01/01-2009/12/31 Val:2010/01/01-2014/12/31 Test:2015/01/01-2024/12/31
Data frequency	Daily returns	Monthly	Monthly
Rebalancing	Every 20 days	Quarterly and semi-annual	Every month
Portfolios created (number of assets considered)	4 portfolios with 10, 20, 40 and 80 most liquid assets	1 with all the stocks	1 (top 20% quantile)
Long-only/Short strategies	Both	Both	Both Mixed

Appendix A

Cumulative returns for both Long-only and Short positions

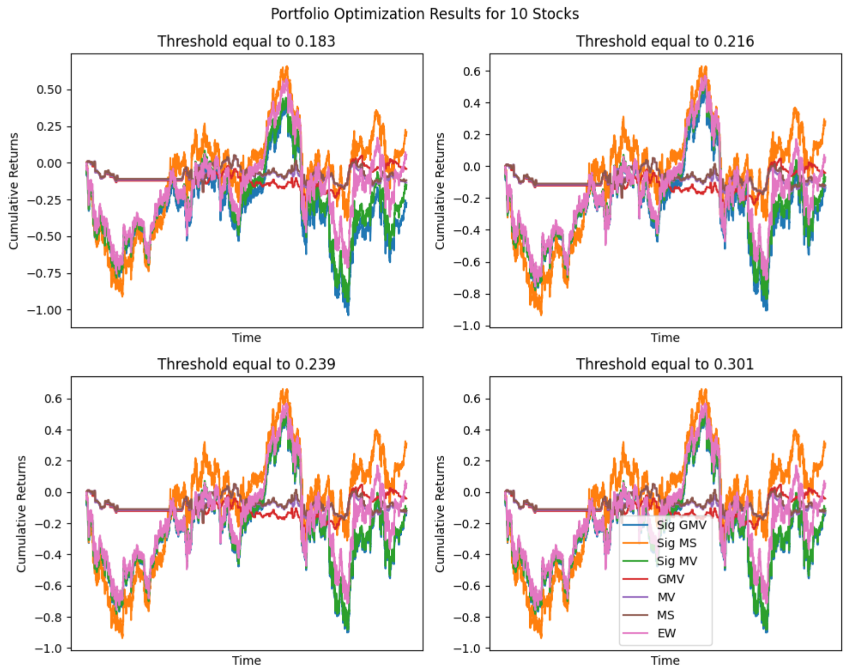

Figure 7. Comparison of the different strategies with long-only position using the 10 most liquid stocks

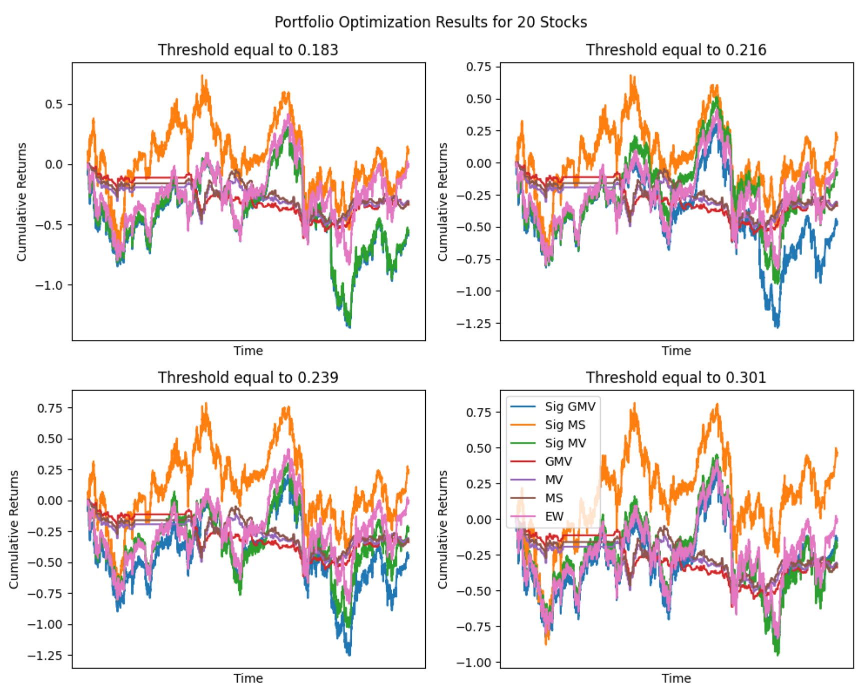

Figure 8. Comparison of the different strategies with long-only position using the 20 most liquid stocks

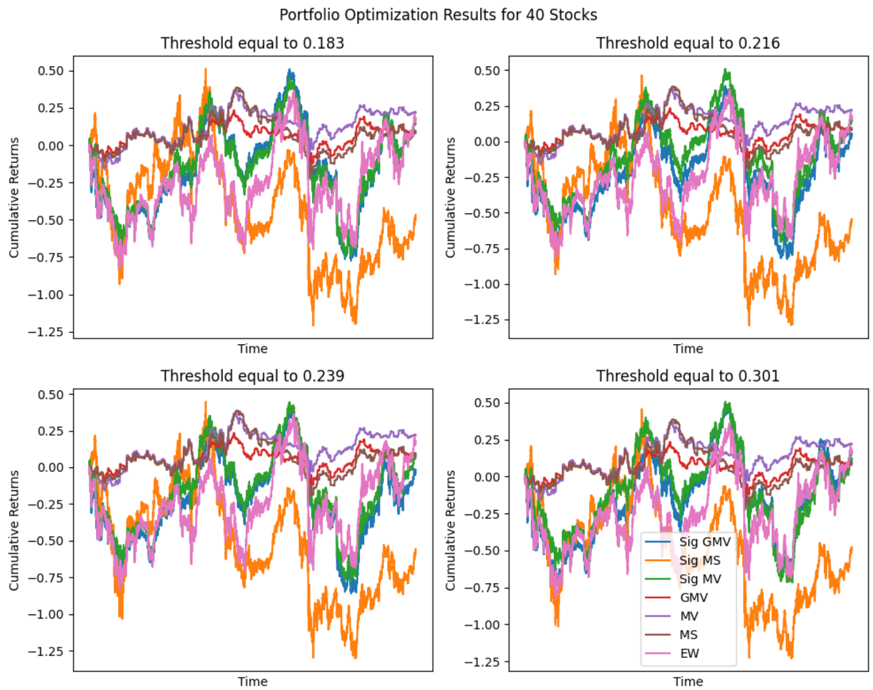

Figure 9. Comparison of the different strategies with long-only position using the 40 most liquid stocks

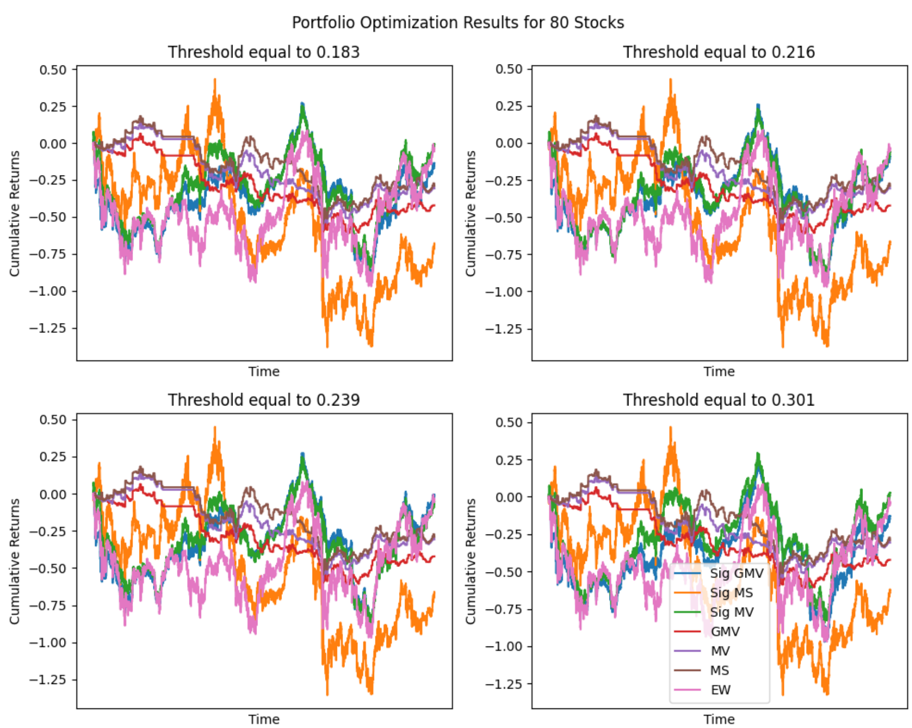

Figure 10. Comparison of the different strategies with long-only position using the 80 most liquid stocks

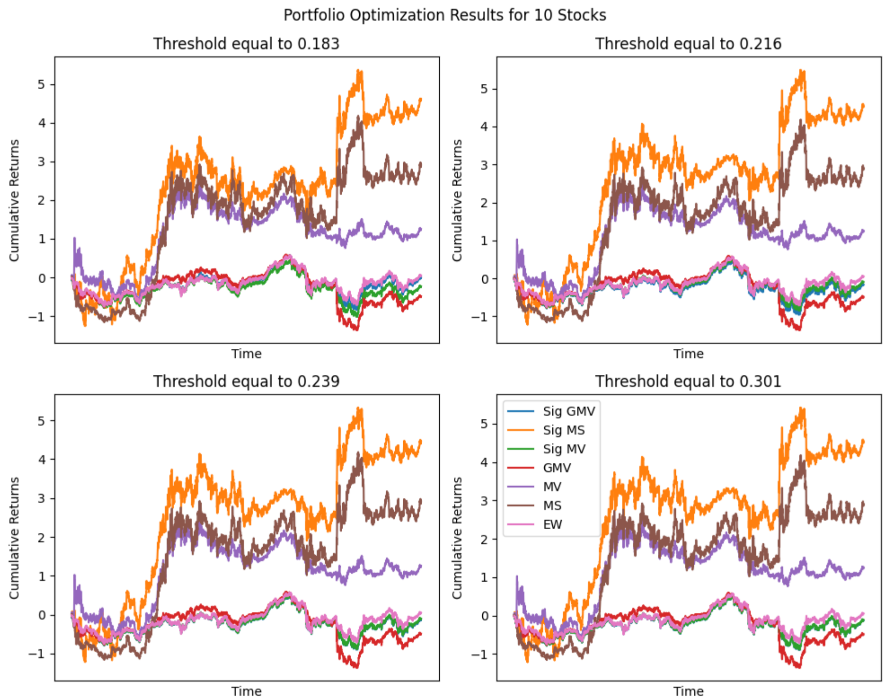

Figure 11. Comparison of the different strategies with short position using the 10 most liquid stocks

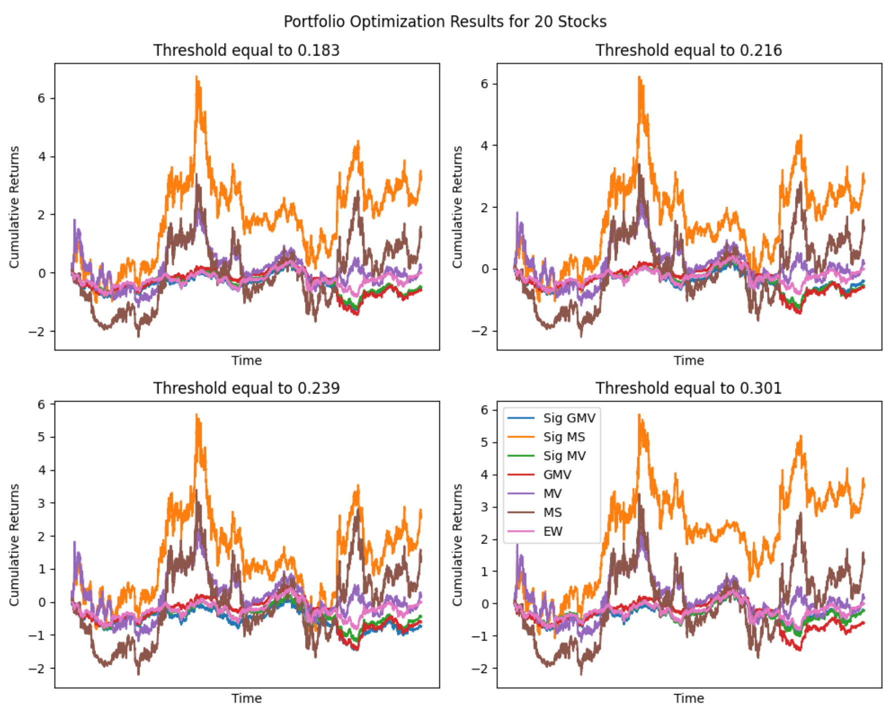

Figure 12. Comparison of the different strategies with short position using the 20 most liquid stocks

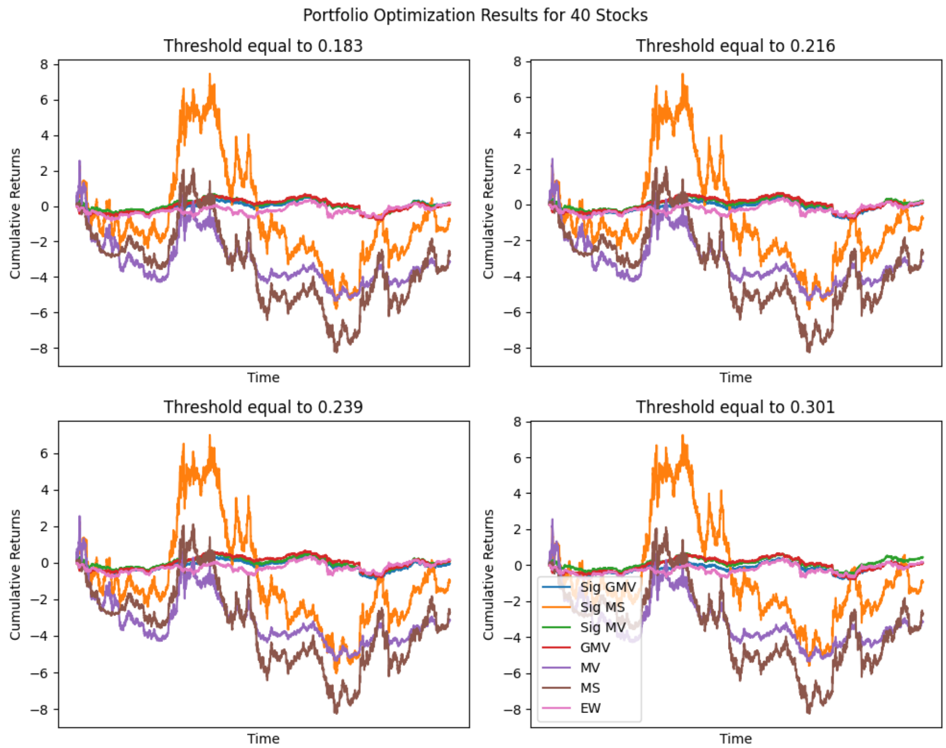

Figure 13. Comparison of the different strategies with short position using the 40 most liquid stocks

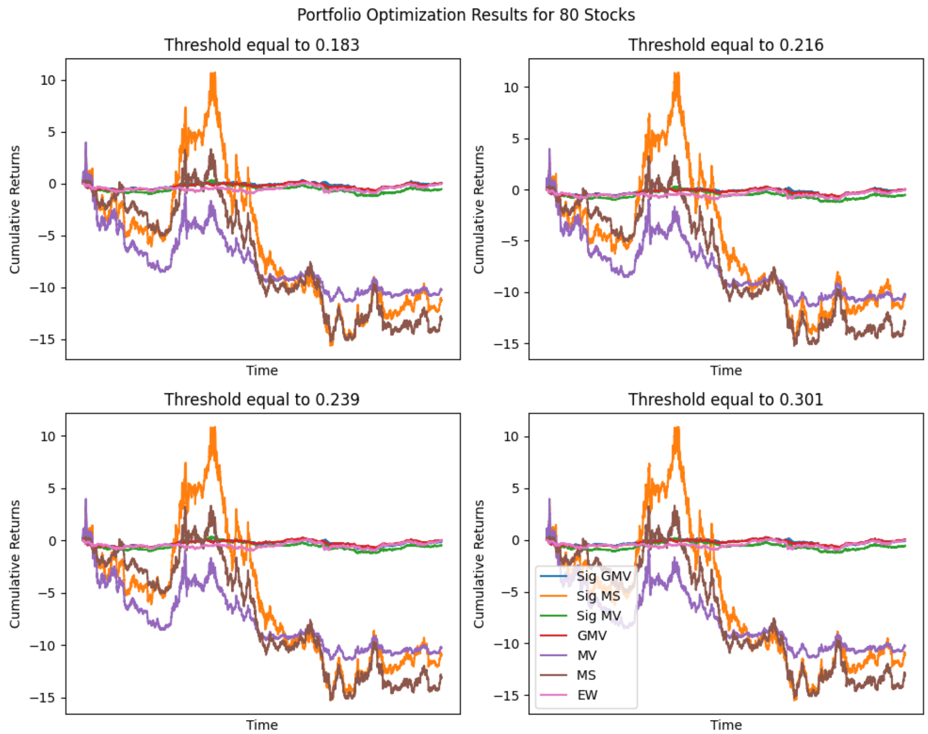

Figure 14. Comparison of the different strategies with short position using the 80 most liquid stocks

Results for train set of Long-only position

Long only, first 10

	Strategy	Theta	Number Asset	Mean Return	Std Return	Excess kurtosis	Skewness	Cumulative Return	Sharpe Ratio	Max Drowdown	CVaR 95%
0	GMV	0	first_10	-0.1518	6.806267	175.94816	-5.10291	-4.14148	-0.31615	-30.2815	-0.60116
4	MV	0	first_10	-0.49383	6.993331	162.72479	-4.37062	-13.496	-0.3566	-24.1568	-0.62763
8	MS	0	first_10	-0.43518	7.013969	146.72417	-3.35478	-11.8897	-0.34719	-22.9911	-0.63567
12	EW	0	first_10	**0.201492**	23.37009	**2.577642**	-0.00918	**5.487554**	**-0.07696**	-78.3016	-3.36854
16	Sig_NGMV	1.96	first_10	-1.0481	22.64827	3.099637	-0.08361	-28.7235	-0.13458	-81.4567	-3.26601
20	Sig_NGMV	2.326	first_10	-0.43259	22.68874	2.944517	-0.05017	-11.8188	-0.10722	-80.0937	-3.27032
24	Sig_NGMV	2.575	first_10	-0.43381	22.70989	2.934111	-0.05256	-11.8521	-0.10717	-80.0955	-3.27444
28	Sig_NGMV	3.291	first_10	-0.44976	22.71277	2.931219	-0.052	-12.2889	-0.10786	-80.2054	-3.27444
32	Sig_NMS	1.96	first_10	**0.893773**	25.57635	**2.109911**	0.081577	**24.25813**	**-0.04325**	-71.1366	-3.60128
36	Sig_NMS	2.326	first_10	**1.179788**	25.47341	**2.139508**	0.06702	**31.97575**	**-0.0322**	-70.9666	-3.59033
40	Sig_NMS	2.575	first_10	**1.164199**	25.47372	**2.136946**	0.06756	**31.55569**	**-0.03281**	-71.0299	-3.59033
44	Sig_NMS	3.291	first_10	**1.148337**	25.47001	**2.140682**	0.067619	**31.12818**	**-0.03344**	-71.0299	-3.58973
48	Sig_NMV	1.96	first_10	-0.46843	22.57341	2.587277	-0.07761	-12.8001	-0.10935	-79.567	-3.25557
52	Sig_NMV	2.326	first_10	-0.40073	22.68535	2.896333	-0.04474	-10.9465	-0.10583	-80.3463	-3.26813
56	Sig_NMV	2.575	first_10	-0.36095	22.69866	2.938888	-0.04717	-9.85794	-0.10401	-80.1287	-3.26909
60	Sig_NMV	3.291	first_10	-0.38929	22.69667	2.941463	-0.04738	-10.6333	-0.10527	-80.1287	-3.26922

Long only, first 20

	Strategy	Theta	Number Asset	Mean Return	Std Return	Excess kurtosis	Skewness	Cumulative Return	Sharpe Ratio	Max Drowdown	CVaR 95%
1	GMV	0	first_20	-1.22872	7.128582	181.7505	-2.53586	-33.7039	-0.45293	-48.3377	-0.69875
5	MV	0	first_20	-1.14558	7.869472	169.9011	-0.56517	-31.4101	-0.39972	-47.1708	-0.74875
9	MS	0	first_20	-1.19919	8.483244	168.0094	-1.40969	-32.8889	-0.37712	-48.3764	-0.81629
13	EW	0	first_20	**0.006686**	24.14536	**2.653485**	-0.04279	**0.182271**	**-0.08256**	-78.0682	-3.48001
17	Sig_NGMV	1.96	first_20	-1.79882	22.01362	4.98874	-0.19804	-49.4838	-0.17257	-84.2032	-3.17612
21	Sig_NGMV	2.326	first_20	-1.40316	22.59865	4.156323	-0.0776	-38.5227	-0.15059	-81.859	-3.24622
25	Sig_NGMV	2.575	first_20	-1.03151	23.06418	4.338586	-0.16272	-28.2663	-0.13144	-83.297	-3.30633
29	Sig_NGMV	3.291	first_20	-0.81065	23.36466	2.66027	-0.00809	-22.1897	-0.1203	-80.1885	-3.35595
33	Sig_NMS	1.96	first_20	**0.407321**	28.15946	**1.890807**	0.074183	**11.08188**	**-0.05656**	-84.7567	-4.00443
37	Sig_NMS	2.326	first_20	**1.040301**	27.70336	**1.605586**	0.068182	**28.21466**	**-0.03464**	-82.567	-3.93259
41	Sig_NMS	2.575	first_20	**1.781714**	27.39206	**1.816661**	0.077613	**48.14697**	**-0.00797**	-79.008	-3.89375
45	Sig_NMS	3.291	first_20	**1.968188**	26.80045	**1.959434**	0.049327	**53.13739**	**-0.00119**	-77.9688	-3.80728
49	Sig_NMV	1.96	first_20	-1.75605	21.91002	4.772653	-0.16137	-48.2968	-0.17143	-83.0221	-3.15019
53	Sig_NMV	2.326	first_20	-1.26276	22.42465	3.808635	-0.11127	-34.6435	-0.1455	-83.2275	-3.21608
57	Sig_NMV	2.575	first_20	-1.15928	23.01503	2.706173	-0.0141	-31.7881	-0.13727	-82.7486	-3.31998
61	Sig_NMV	3.291	first_20	-0.1723	23.43441	3.122347	-0.03783	-4.70114	-0.0927	-79.6003	-3.37741

Long only, first 40

	Strategy	Theta	Number Asset	Mean Return	Std Return	Excess kurtosis	Skewness	Cumulative Return	Sharpe Ratio	Max Drowdown	CVaR 95%
2	GMV	0	first_40	0.328041	7.916456	139.44387	-0.86888	8.928444	-0.2112	-33.589	-0.75075
6	MV	0	first_40	**0.795727**	8.721234	142.97303	0.550552	**21.60753**	-0.13809	-37.6418	-0.80589
10	MS	0	first_40	0.291808	8.909704	110.46652	0.201304	7.943697	-0.19172	-46.9369	-0.87134
14	EW	0	first_40	0.655619	23.94515	**3.169105**	-0.10104	17.81533	**-0.05614**	-74.0072	-3.49194
18	Sig_NGMV	1.96	first_40	**0.571288**	19.46871	5.315499	-0.16828	**15.53026**	**-0.07339**	-74.5408	-2.84721
22	Sig_NGMV	2.326	first_40	0.566264	19.52384	5.612748	-0.19969	15.39407	-0.07344	-75.7393	-2.86229
26	Sig_NGMV	2.575	first_40	0.005514	19.47773	5.327948	-0.16953	0.15032	-0.1024	-74.7896	-2.85206
30	Sig_NGMV	3.291	first_40	-0.20501	20.00227	4.46275	-0.10946	-5.59454	-0.11024	-75.2776	-2.93164
34	Sig_NMS	1.96	first_40	-1.95694	27.97773	**1.51031**	0.062567	-53.8767	-0.14143	-91.0477	-4.01391
38	Sig_NMS	2.326	first_40	-1.97337	27.88869	**1.482793**	0.060757	-54.3336	-0.14247	-91.1368	-4.00338
42	Sig_NMS	2.575	first_40	-1.9715	27.33203	**1.231486**	0.022902	-54.2817	-0.14531	-90.689	-3.92392
46	Sig_NMS	3.291	first_40	-1.756	26.9887	**1.65728**	0.042973	-48.2955	-0.13917	-89.9478	-3.87414
50	Sig_NMV	1.96	first_40	0.520489	19.51399	5.171633	-0.13773	14.15286	-0.07582	-74.1998	-2.8458
54	Sig_NMV	2.326	first_40	**0.785671**	19.55498	5.11727	-0.1063	**21.33551**	**-0.0621**	-75.3973	-2.85071
58	Sig_NMV	2.575	first_40	**0.663272**	19.50399	4.773527	-0.11743	**18.02259**	**-0.06854**	-75.0861	-2.83706
62	Sig_NMV	3.291	first_40	**0.336153**	19.95492	4.46526	-0.11934	**9.148874**	**-0.08338**	-75.1616	-2.91335

Long only, first 80

	Strategy	Theta	Number Asset	Mean Return	Std Return	Excess kurtosis	Skewness	Cumulative Return	Sharpe Ratio	Max Drowdown	CVaR 95%
3	GMV	0	first_80	-1.53669	7.371922	182.3733	-3.4847	-42.2169	-0.47975	-52.7564	-0.74856
7	MV	0	first_80	-1.12159	7.891195	156.8635	-1.82247	-30.7487	-0.39558	-54.1896	-0.76733
11	MS	0	first_80	-1.06072	8.240372	144.629	-1.80543	-29.071	-0.37143	-54.2758	-0.82334
15	EW	0	first_80	**-0.13864**	22.85573	**3.791284**	-0.12184	**-3.78209**	**-0.09357**	-79.0812	-3.34615
19	Sig_NGMV	1.96	first_80	-0.33209	17.73855	7.332695	-0.13558	-9.06838	-0.13147	-74.1026	-2.5544
23	Sig_NGMV	2.326	first_80	-0.04872	17.71669	7.759767	-0.14832	-1.32849	-0.11564	-72.7417	-2.55861
27	Sig_NGMV	2.575	first_80	-0.53274	17.79133	7.525484	-0.14262	-14.5622	-0.14236	-74.0522	-2.56079
31	Sig_NGMV	3.291	first_80	-0.26851	17.74928	8.071375	-0.16725	-7.3298	-0.12781	-72.8031	-2.55175
35	Sig_NMS	1.96	first_80	-2.30416	27.04665	**1.330391**	0.000102	-63.5477	-0.15914	-90.9916	-3.91184
39	Sig_NMS	2.326	first_80	-2.31921	27.0303	**1.330266**	-0.00172	-63.9679	-0.15979	-91.2909	-3.90918
43	Sig_NMS	2.575	first_80	-2.13262	27.016	**1.291249**	0.002619	-58.7658	-0.15297	-90.7109	-3.9116
47	Sig_NMS	3.291	first_80	-2.49354	27.02482	**1.325985**	-0.0071	-68.8371	-0.16628	-91.2299	-3.91544
51	Sig_NMV	1.96	first_80	**0.205151**	18.10129	6.416965	-0.10957	**5.587113**	**-0.09916**	-70.8707	-2.61436
55	Sig_NMV	2.326	first_80	**0.032465**	18.07065	6.388944	-0.10427	**0.884927**	**-0.10888**	-71.797	-2.60827
59	Sig_NMV	2.575	first_80	**0.131247**	18.069	6.467232	-0.10409	**3.575697**	**-0.10342**	-71.2443	-2.60423
63	Sig_NMV	3.291	first_80	**-0.09447**	18.06666	6.626879	-0.0956	**-2.57671**	**-0.11593**	-73.0392	-2.60288

Results for test set of Long-only position

First 10

	Strategy	Theta	Number Asset	Mean Return	Std Return	Excess kurtosis	Skewness	Cumulative Return	Sharpe Ratio	Max Drowdown	CVaR 95%
0	GMV	0	first_10	-0.1518	6.806267	175.9482	-5.10291	-4.14148	-0.31615	-30.2815	-0.60116
4	MV	0	first_10	-0.49383	6.993331	162.7248	-4.37062	-13.496	-0.3566	-24.1568	-0.62763
8	MS	0	first_10	-0.43518	7.013969	146.7242	-3.35478	-11.8897	-0.34719	-22.9911	-0.63567
12	EW	0	first_10	**0.201492**	23.37009	**2.577642**	-0.00918	**5.487554**	**-0.07696**	-78.3016	-3.36854
16	Sig_NGMV	1.96	first_10	-0.52019	22.75091	2.805019	-0.05542	-14.2181	-0.11077	-81.104	-3.28035
20	Sig_NGMV	2.326	first_10	-0.4711	22.72885	2.934875	-0.05305	-12.8732	-0.10872	-80.2314	-3.27721
24	Sig_NGMV	2.575	first_10	-0.44455	22.71634	2.926222	-0.05222	-12.1463	-0.10761	-80.1989	-3.27574
28	Sig_NGMV	3.291	first_10	-0.44976	22.71277	2.931219	-0.052	-12.2889	-0.10786	-80.2054	-3.27444
32	Sig_NMS	1.96	first_10	**1.061846**	25.52799	**2.092536**	0.052915	**28.79594**	**-0.03675**	-71.3312	-3.60543
36	Sig_NMS	2.326	first_10	**1.108044**	25.41003	**1.975112**	0.063891	**30.04193**	**-0.0351**	-71.5077	-3.57912
40	Sig_NMS	2.575	first_10	**1.148337**	25.47001	**2.140682**	0.067619	**31.12818**	**-0.03344**	-71.0299	-3.58973
44	Sig_NMS	3.291	first_10	**1.148337**	25.47001	**2.140682**	0.067619	**31.12818**	**-0.03344**	-71.0299	-3.58973
48	Sig_NMV	1.96	first_10	-1.06543	22.75356	3.057551	-0.05855	-29.2008	-0.13472	-82.9486	-3.27557
52	Sig_NMV	2.326	first_10	-0.35037	22.70421	2.931848	-0.04745	-9.56842	-0.10352	-80.2343	-3.2702
56	Sig_NMV	2.575	first_10	-0.38055	22.69766	2.940078	-0.04747	-10.3942	-0.10488	-80.2053	-3.26922
60	Sig_NMV	3.291	first_10	-0.40095	22.69795	2.940284	-0.04731	-10.9527	-0.10578	-80.1287	-3.26974

First 20

	Strategy	Theta	Number Asset	Mean Return	Std Return	Excess kurtosis	Skewness	Cumulative Return	Sharpe Ratio	Max Drowdown	CVaR 95%
1	GMV	0	first_20	-1.22872	7.128582	181.7505	-2.535863	-33.7039	-0.45293	-48.3377	-0.69875
5	MV	0	first_20	-1.14558	7.869472	169.9011	-0.565168	-31.4101	-0.39972	-47.1708	-0.74875
9	MS	0	first_20	-1.19919	8.483244	168.0094	-1.409691	-32.8889	-0.37712	-48.3764	-0.81629
13	EW	0	first_20	**0.006686**	24.14536	**2.653485**	-0.042789	**0.182271**	**-0.08256**	-78.0682	-3.48001
17	Sig_NGMV	1.96	first_20	-1.45981	21.95224	4.876787	-0.171878	-40.0893	-0.15761	-83.3517	-3.15932
21	Sig_NGMV	2.326	first_20	-1.62915	22.293	3.210699	-0.098283	-44.7781	-0.16279	-82.2796	-3.22345
25	Sig_NGMV	2.575	first_20	-0.01404	23.0451	3.304734	-0.088218	-0.38289	-0.0874	-77.3281	-3.30445
29	Sig_NGMV	3.291	first_20	-0.70672	23.49253	2.779815	-0.097662	-19.3347	-0.11522	-81.3178	-3.38562
33	Sig_NMS	1.96	first_20	**0.457859**	28.09332	**1.501995**	0.087529	**12.45372**	**-0.05489**	-84.5203	-3.98611
37	Sig_NMS	2.326	first_20	**1.237402**	27.7609	**1.852144**	0.091622	**33.52776**	**-0.02747**	-82.3806	-3.94474
41	Sig_NMS	2.575	first_20	**1.262676**	27.41516	**1.670293**	0.057382	**34.20829**	**-0.0269**	-81.5817	-3.89459
45	Sig_NMS	3.291	first_20	**1.506697**	26.94016	**1.95985**	0.049134	**40.77028**	**-0.01831**	-79.0996	-3.83641
49	Sig_NMV	1.96	first_20	-1.73788	21.93109	4.179759	-0.137622	-47.7927	-0.17044	-83.8319	-3.16033
53	Sig_NMV	2.326	first_20	-0.56324	22.39057	3.337706	-0.058092	-15.3982	-0.11448	-79.3593	-3.20303
57	Sig_NMV	2.575	first_20	-1.96561	23.06725	2.919091	-0.037453	-54.1177	-0.17192	-85.1364	-3.32171
61	Sig_NMV	3.291	first_20	-0.58948	23.41197	2.96814	-0.057171	-16.1177	-0.11061	-80.3263	-3.36734

First 40

	Strategy	Theta	Number Asset	Mean Return	Std Return	Excess kurtosis	Skewness	Cumulative Return	Sharpe Ratio	Max Drowdown	CVaR 95%
2	GMV	0	first_40	0.328041	7.916456	139.4439	-0.86888	8.928444	-0.2112	-33.589	-0.75075
6	MV	0	first_40	**0.795727**	8.721234	142.973	0.550552	**21.60753**	**-0.13809**	-37.6418	-0.80589
10	MS	0	first_40	-0.6012	6.398123	117.7638	-0.29755	-16.4392	-0.40656	-41.3105	-0.61278
14	EW	0	first_40	0.655619	23.94515	**3.169105**	-0.10104	17.81533	-0.05614	-74.0072	-3.49194
18	Sig_NGMV	1.96	first_40	-0.19446	19.53293	5.268655	-0.1698	-5.30648	-0.11235	-76.8275	-2.86771
22	Sig_NGMV	2.326	first_40	0.270464	19.55261	5.661749	-0.18472	7.363459	-0.08846	-75.3531	-2.86011
26	Sig_NGMV	2.575	first_40	0.313196	19.55792	5.583217	-0.18764	8.525042	-0.08625	-75.2599	-2.85901
30	Sig_NGMV	3.291	first_40	0.379441	19.99423	4.17596	-0.10223	10.32477	-0.08105	-73.0465	-2.92516
34	Sig_NMS	1.96	first_40	-2.03886	27.83885	**1.404626**	0.038724	-56.1554	-0.14508	-91.0849	-3.99682
38	Sig_NMS	2.326	first_40	-2.05778	27.76583	**1.396218**	0.039014	-56.6818	-0.14614	-91.2233	-3.98741
42	Sig_NMS	2.575	first_40	-2.24162	27.83583	**1.349142**	0.048605	-61.8035	-0.15238	-91.6717	-3.99133
46	Sig_NMS	3.291	first_40	-1.75347	26.46475	**1.245858**	0.041254	-48.2254	-0.14183	-89.6925	-3.79159
50	Sig_NMV	1.96	first_40	**0.349909**	19.52398	5.291025	-0.1321	**9.522597**	**-0.08452**	-73.7191	-2.85124
54	Sig_NMV	2.326	first_40	**0.248544**	19.56131	5.355722	-0.1511	**6.767412**	**-0.08954**	-75.797	-2.85368
58	Sig_NMV	2.575	first_40	**0.882932**	19.50175	5.080198	-0.13497	**23.96517**	**-0.05728**	-74.0647	-2.84205
62	Sig_NMV	3.291	first_40	**0.464302**	20.05822	4.539793	-0.09	**12.62859**	**-0.07656**	-75.4328	-2.91869

First 80

	Strategy	Theta	Number Asset	Mean Return	Std Return	Excess kurtosis	Skewness	Cumulative Return	Sharpe Ratio	Max Drowdown	CVaR 95%
3	GMV	0	first_80	-1.53669	7.371922	182.3733	-3.484702	-42.2169	-0.47975	-52.7564	-0.74856
7	MV	0	first_80	-1.12159	7.891195	156.8635	-1.822471	-30.7487	-0.39558	-54.1896	-0.76733
11	MS	0	first_80	-1.06072	8.240372	144.629	-1.805425	-29.071	-0.37143	-54.2758	-0.82334
15	EW	0	first_80	**-0.13864**	22.85573	**3.791284**	-0.121843	**-3.78209**	**-0.09357**	-79.0812	-3.34615
19	Sig_NGMV	1.96	first_80	-0.28402	17.79494	7.977828	-0.155523	-7.75402	-0.12835	-72.8779	-2.56499
23	Sig_NGMV	2.326	first_80	-0.60691	17.80577	7.130282	-0.139958	-16.5958	-0.14641	-73.8825	-2.57265
27	Sig_NGMV	2.575	first_80	-0.10704	17.76977	7.74524	-0.173052	-2.91959	-0.11857	-73.2499	-2.5679
31	Sig_NGMV	3.291	first_80	-0.62428	17.81632	7.696864	-0.165784	-17.0723	-0.1473	-73.7525	-2.57453
35	Sig_NMS	1.96	first_80	-3.79245	27.32348	**2.484428**	0.163859	-105.393	-0.212	-93.1076	-3.91752
39	Sig_NMS	2.326	first_80	-3.63185	27.26341	**2.41576**	0.155874	-100.847	-0.20657	-92.745	-3.90932
43	Sig_NMS	2.575	first_80	-3.50617	27.32703	**2.470293**	0.156084	-97.2939	-0.20149	-92.7751	-3.91848
47	Sig_NMS	3.291	first_80	-3.65376	27.25778	**2.417466**	0.155206	-101.466	-0.20742	-92.9713	-3.90859
51	Sig_NMV	1.96	first_80	**-0.02563**	18.02434	6.308338	-0.061663	**-0.69886**	**-0.11238**	-71.644	-2.59614
55	Sig_NMV	2.326	first_80	**-0.02862**	18.05989	6.053507	-0.066371	**-0.78042**	**-0.11233**	-72.572	-2.60863
59	Sig_NMV	2.575	first_80	**0.283857**	18.06766	6.154846	-0.051001	**7.72755**	**-0.09498**	-71.2607	-2.60553
63	Sig_NMV	3.291	first_80	**0.131286**	18.07066	6.260203	-0.06895	**3.576761**	**-0.10341**	-71.488	-2.60706

Results for train set of Short position

First 10

	Strategy	Theta	Number Asset	Mean Return	Std Return	Excess kurtosis	Skewness	Cumulative Return	Sharpe Ratio	Max Drowdown	CVaR 95%
0	GMV	0	first_10	-1.76548	22.337574	9.073046	-0.44464	-48.5587	-0.16857	-88.9425	-3.2522
4	MV	0	first_10	4.711737	47.242711	4.761975	-0.09154	125.5281	0.0574	-96.7036	-7.00547
8	MS	0	first_10	**11.27345**	69.201337	2.866143	-0.23436	**291.2747**	**0.134007**	-99.7736	-10.4497
12	EW	0	first_10	0.201492	23.370086	**2.577642**	-0.00918	5.487554	-0.07696	-78.3016	-3.36854
16	Sig_NGMV	1.96	first_10	0.002471	22.765134	3.014587	-0.06665	0.067359	-0.08775	-80.6616	-3.28257
20	Sig_NGMV	2.326	first_10	-0.58797	22.742555	3.252009	-0.0808	-16.0763	-0.11379	-80.6376	-3.27786
24	Sig_NGMV	2.575	first_10	-0.44463	22.713329	2.930575	-0.05205	-12.1484	-0.10763	-80.2054	-3.27422
28	Sig_NGMV	3.291	first_10	-0.44998	22.712714	2.931279	-0.052	-12.295	-0.10787	-80.2054	-3.27445
32	Sig_NMS	1.96	first_10	**18.42698**	72.122057	**2.474274**	-0.19442	**461.2255**	**0.227766**	-99.2954	-10.7371
36	Sig_NMS	2.326	first_10	**18.19154**	72.589459	**2.653078**	-0.209	**455.7966**	**0.223056**	-99.3401	-10.8689
40	Sig_NMS	2.575	first_10	**17.74407**	72.732181	**2.578361**	-0.19931	**445.4491**	**0.216466**	-99.4521	-10.8591
44	Sig_NMS	3.291	first_10	**18.15241**	72.766298	**2.572347**	-0.20043	**454.8932**	**0.221976**	-99.402	-10.8693
48	Sig_NMV	1.96	first_10	-0.83105	22.685095	2.643158	-0.04767	-22.7503	-0.1248	-81.3815	-3.27618
52	Sig_NMV	2.326	first_10	-0.33032	22.680041	3.062348	-0.02634	-9.02007	-0.10275	-79.5555	-3.2623
56	Sig_NMV	2.575	first_10	-0.35423	22.70134	2.940944	-0.04902	-9.67413	-0.10371	-80.1287	-3.27228
60	Sig_NMV	3.291	first_10	-0.41205	22.699299	2.941422	-0.04838	-11.2563	-0.10626	-80.1287	-3.27164

First 20

	Strategy	Theta	Number Asset	Mean Return	Std Return	Excess kurtosis	Skewness	Cumulative Return	Sharpe Ratio	Max Drowdown	CVaR 95%
1	GMV	0	first_20	-2.15865	22.3535	9.041079	-0.51485	-59.4908	-0.18604	-88.7847	-3.25736
5	MV	0	first_20	0.66505	62.433837	3.778745	-0.25363	18.07075	-0.02138	-99.931	-9.53992
9	MS	0	first_20	**4.863517**	93.851502	**1.424333**	-0.23279	**129.4776**	**0.030511**	-99.9994	-14.1723
13	EW	0	first_20	0.006686	24.145358	2.653485	-0.04279	0.182271	-0.08256	-78.0682	-3.48001
17	Sig_NGMV	1.96	first_20	-1.90491	21.837003	4.113671	-0.13809	-52.4304	-0.17882	-83.6141	-3.15334
21	Sig_NGMV	2.326	first_20	-1.43653	22.468535	3.909908	-0.09371	-39.4455	-0.15295	-84.1834	-3.20945
25	Sig_NGMV	2.575	first_20	-2.64071	23.065632	3.65703	-0.07121	-72.9544	-0.2012	-87.2221	-3.31386
29	Sig_NGMV	3.291	first_20	-0.74941	23.498209	3.387053	-0.09145	-20.5069	-0.11701	-79.6248	-3.38828
33	Sig_NMS	1.96	first_20	**12.66207**	106.27315	**1.882803**	-0.24964	**325.1005**	**0.100327**	-100	-16.0705
37	Sig_NMS	2.326	first_20	**10.90648**	104.28077	**1.788404**	-0.26095	**282.2655**	**0.085409**	-100	-15.7887
41	Sig_NMS	2.575	first_20	**10.01748**	101.41059	**1.866069**	-0.18784	**260.3163**	**0.07906**	-99.9999	-15.2725
45	Sig_NMS	3.291	first_20	**14.33046**	94.846797	**1.460346**	-0.2541	**365.1962**	**0.130004**	-99.9983	-14.3351
49	Sig_NMV	1.96	first_20	-1.74673	22.005156	4.600423	-0.14013	-48.0382	-0.17027	-84.4453	-3.16746
53	Sig_NMV	2.326	first_20	-1.96177	22.608781	3.958509	-0.1695	-54.0109	-0.17523	-85.2398	-3.27339
57	Sig_NMV	2.575	first_20	-1.58049	23.061527	2.081278	-0.04868	-43.4299	-0.15526	-84.479	-3.32713
61	Sig_NMV	3.291	first_20	-0.78606	23.437178	3.033069	-0.04832	-21.5138	-0.11887	-81.4763	-3.37584

First 40

	Strategy	Theta	Number Asset	Mean Return	Std Return	Excess kurtosis	Skewness	Cumulative Return	Sharpe Ratio	Max Drowdown	CVaR 95%
2	GMV	0	first_40	**0.397608**	19.920454	6.803734	-0.47639	10.81815	-0.08044	-80.3278	-2.92867
6	MV	0	first_40	-10.9208	86.562872	7.767715	-0.58333	-315.195	-0.14926	-100	-13.7308
10	MS	0	first_40	-9.49983	117.98822	**3.099351**	-0.45579	-272.07	**-0.09747**	-100	-18.0671
14	EW	0	first_40	0.655619	23.945146	3.169105	-0.10104	**17.81533**	-0.05614	-74.0072	-3.49194
18	Sig_NGMV	1.96	first_40	**0.709734**	19.411186	4.575879	-0.14081	**19.28065**	-0.06647	-74.9512	-2.80241
22	Sig_NGMV	2.326	first_40	0.28141	19.377809	4.584558	-0.15068	7.66105	-0.08869	-75.7473	-2.80653
26	Sig_NGMV	2.575	first_40	-0.18317	19.409538	4.921399	-0.15635	-4.99814	-0.11248	-77.7437	-2.81379
30	Sig_NGMV	3.291	first_40	0.536002	19.817232	4.452916	-0.12331	14.57358	-0.07388	-73.9471	-2.88089
34	Sig_NMS	1.96	first_40	-2.9018	147.41603	**2.888823**	-0.436	-80.2743	**-0.03325**	-100	-22.8826
38	Sig_NMS	2.326	first_40	-2.79624	147.60782	**3.305828**	-0.48827	-77.3125	**-0.03249**	-100	-22.8596
42	Sig_NMS	2.575	first_40	-3.67988	147.16298	**3.482687**	-0.50084	-102.205	**-0.0386**	-100	-22.8319
46	Sig_NMS	3.291	first_40	-3.36222	147.13342	**3.418912**	-0.51123	-93.2305	-0.03645	-100	-22.8689
50	Sig_NMV	1.96	first_40	0.69658	21.387326	14.32712	0.015904	18.92453	-0.06094	-80.446	-3.05678
54	Sig_NMV	2.326	first_40	**0.827052**	21.376362	13.99231	0.004035	**22.45465**	-0.05487	-79.4312	-3.0614
58	Sig_NMV	2.575	first_40	**0.374633**	21.329545	13.90561	0.000111	**10.1942**	-0.0762	-80.8911	-3.05272
62	Sig_NMV	3.291	first_40	**1.634157**	21.131519	11.97952	0.046612	**44.19157**	**-0.01731**	-75.4032	-3.01881

First 80

	Strategy	Theta	Number Asset	Mean Return	Std Return	Excess kurtosis	Skewness	Cumulative Return	Sharpe Ratio	Max Drowdown	CVaR 95%
3	GMV	0	first_80	**0.152405**	17.783012	4.925048	-0.3051	**4.151713**	-0.1039	-65.8143	-2.55925
7	MV	0	first_80	-31.3629	125.11554	9.90855	-0.6753	-1025.2	-0.26666	-100	-20.2185
11	MS	0	first_80	-38.1448	155.95684	**2.236931**	-0.43108	-1308.34	-0.25741	-100	-23.8392
15	EW	0	first_80	-0.13864	22.855727	3.791284	-0.12184	-3.78209	**-0.09357**	-79.0812	-3.34615
19	Sig_NGMV	1.96	first_80	**0.062802**	19.042849	**4.835184**	-0.03186	**1.711561**	**-0.10173**	-71.6844	-2.71198
23	Sig_NGMV	2.326	first_80	**-0.03961**	19.023384	**4.394498**	-0.02731	**-1.08006**	**-0.10722**	-71.2658	-2.70704
27	Sig_NGMV	2.575	first_80	**-0.41055**	19.074772	4.921835	-0.02984	**-11.2154**	**-0.12637**	-72.3776	-2.71724
31	Sig_NGMV	3.291	first_80	**-0.26026**	19.023493	4.593695	-0.02413	**-7.10434**	**-0.11881**	-72.2815	-2.71198
35	Sig_NMS	1.96	first_80	-33.8245	213.41342	4.897652	-0.38205	-1124.61	-0.16786	-100	-33.2718
39	Sig_NMS	2.326	first_80	-31.8395	213.69027	4.883249	-0.35491	-1044.17	-0.15836	-100	-33.257
43	Sig_NMS	2.575	first_80	-33.0153	214.03261	**4.907856**	-0.36761	-1091.53	-0.1636	-100	-33.3157
47	Sig_NMS	3.291	first_80	-33.381	212.23172	**4.414171**	-0.3517	-1106.43	-0.16671	-100	-32.9957
51	Sig_NMV	1.96	first_80	-1.95381	29.621132	84.52505	-0.70005	-53.7897	-0.13348	-94.7779	-4.02617
55	Sig_NMV	2.326	first_80	-1.86688	29.627025	84.90789	-0.57106	-51.3738	-0.13052	-94.6778	-4.03212
59	Sig_NMV	2.575	first_80	-1.61195	29.637526	85.89395	-0.69611	-44.3014	-0.12187	-94.5453	-4.0347
63	Sig_NMV	3.291	first_80	-2.03272	29.705455	85.66326	-0.6781	-55.9844	-0.13576	-94.7828	-4.03714

Results for test set of Short position

First 10

	Strategy	Theta	Number Asset	Mean Return	Std Return	Excess kurtosis	Skewness	Cumulative Return	Sharpe Ratio	Max Drowdown	CVaR 95%
0	GMV	0	first_10	-1.76548	22.337574	9.073046	-0.44464	-48.5587	-0.16857	-88.9425	-3.2522
4	MV	0	first_10	4.711737	47.242711	4.761975	-0.09154	125.5281	0.0574	-96.7036	-7.00547
8	MS	0	first_10	**11.27345**	69.201337	2.866143	-0.23436	**291.2747**	**0.134007**	-99.7736	-10.4497
12	EW	0	first_10	0.201492	23.370086	**2.577642**	-0.00918	5.487554	-0.07696	-78.3016	-3.36854
16	Sig_NGMV	1.96	first_10	-0.26298	22.713871	3.109191	-0.04498	-7.17867	-0.09963	-78.0168	-3.27108
20	Sig_NGMV	2.326	first_10	-0.46442	22.678065	2.907146	-0.04635	-12.6904	-0.10867	-80.3515	-3.26465
24	Sig_NGMV	2.575	first_10	-0.39349	22.714695	2.930836	-0.05119	-10.7484	-0.10537	-80.2054	-3.27489
28	Sig_NGMV	3.291	first_10	-0.44998	22.712714	2.931279	-0.052	-12.295	-0.10787	-80.2054	-3.27445
32	Sig_NMS	1.96	first_10	**13.87465**	72.051593	**2.762562**	-0.19813	**354.3001**	**0.164808**	-99.6959	-10.7769
36	Sig_NMS	2.326	first_10	**18.43548**	72.712738	**2.59008**	-0.19767	**461.4213**	**0.226033**	-99.3502	-10.8686
40	Sig_NMS	2.575	first_10	**17.98587**	72.756931	**2.576525**	-0.20019	**451.0455**	**0.219716**	-99.402	-10.8693
44	Sig_NMS	3.291	first_10	**18.15241**	72.766298	**2.572347**	-0.20043	**454.8932**	**0.221976**	-99.402	-10.8693
48	Sig_NMV	1.96	first_10	-0.68087	22.688138	2.968558	-0.05083	-18.6251	-0.11816	-80.4585	-3.2689
52	Sig_NMV	2.326	first_10	-0.48243	22.68402	2.950379	-0.04691	-13.1836	-0.10944	-80.2565	-3.27018
56	Sig_NMV	2.575	first_10	-0.41205	22.699299	2.941422	-0.04838	-11.2563	-0.10626	-80.1287	-3.27164
60	Sig_NMV	3.291	first_10	-0.41205	22.699299	2.941422	-0.04838	-11.2563	-0.10626	-80.1287	-3.27164

First 20

	Strategy	Theta	Number Asset	Mean Return	Std Return	Excess kurtosis	Skewness	Cumulative Return	Sharpe Ratio	Max Drowdown	CVaR 95%
1	GMV	0	first_20	-2.15865	22.3535	9.041079	-0.51485	-59.4908	-0.18604	-88.7847	-3.25736
5	MV	0	first_20	0.66505	62.433837	3.778745	-0.25363	18.07075	-0.02138	-99.931	-9.53992
9	MS	0	first_20	**4.863517**	93.851502	**1.424333**	-0.23279	**129.4776**	**0.030511**	-99.9994	-14.1723
13	EW	0	first_20	0.006686	24.145358	2.653485	-0.04279	0.182271	-0.08256	-78.0682	-3.48001
17	Sig_NGMV	1.96	first_20	-1.355591	21.903102	4.618789	-0.1502	-37.2078	-0.1532	-82.5176	-3.15267
21	Sig_NGMV	2.326	first_20	-1.185747	22.387151	3.44566	-0.08928	-32.5181	-0.1423	-81.0553	-3.21553
25	Sig_NGMV	2.575	first_20	-0.84105	23.157719	3.566244	-0.08156	-23.0252	-0.12268	-81.2663	-3.32399
29	Sig_NGMV	3.291	first_20	-1.026189	23.4164	2.737442	-0.04407	-28.1198	-0.12923	-81.9928	-3.37897
33	Sig_NMS	1.96	first_20	**13.028257**	106.01008	**1.789219**	-0.24292	**333.9513**	**0.10403**	-100	-16.0042
37	Sig_NMS	2.326	first_20	**11.065606**	104.24548	**1.850654**	-0.24951	**286.1757**	**0.086964**	-100	-15.7802
41	Sig_NMS	2.575	first_20	**10.020389**	102.06037	**1.931175**	-0.26625	**260.3883**	**0.078585**	-99.9999	-15.4296
45	Sig_NMS	3.291	first_20	**6.112188**	93.854322	**1.796913**	-0.25925	**161.755**	**0.043815**	-99.9997	-14.1471
49	Sig_NMV	1.96	first_20	-2.122906	22.044718	5.310084	-0.19988	-58.4951	-0.18703	-86.1748	-3.17421
53	Sig_NMV	2.326	first_20	-1.602793	22.454767	3.275449	-0.0655	-44.0477	-0.16045	-83.9356	-3.21924
57	Sig_NMV	2.575	first_20	-1.583733	23.146807	3.663019	-0.05001	-43.5197	-0.15483	-82.9388	-3.33507
61	Sig_NMV	3.291	first_20	-0.639972	23.358038	2.865243	-0.03799	-17.5027	-0.11302	-80.7152	-3.36199

	Strategy	**Theta**	**Number Asset**	**Mean Return**	**Std Return**	**Excess kurtosis**	**Skewness**	**Cumulative Return**	**Sharpe Ratio**	**Max Drowdown**	**CVaR 95%**
2	GMV	0	first_40	0.397608	19.920454	6.803734	-0.47639	10.81815	-0.08044	-80.3278	-2.92867
6	MV	0	first_40	-10.9208	86.562872	7.767715	-0.58333	-315.195	-0.14926	-100	-13.7308
10	MS	0	first_40	**15.84893**	134.87011	**2.284691**	-0.10918	**401.1856**	**0.102683**	-100	-19.6014
14	EW	0	first_40	0.655619	23.945146	3.169105	-0.10104	17.81533	-0.05614	-74.0072	-3.49194
18	Sig_NGMV	1.96	first_40	0.406882	19.38874	4.715327	-0.15732	11.06996	-0.08217	-75.084	-2.78984
22	Sig_NGMV	2.326	first_40	0.478094	19.406894	4.463302	-0.16318	13.00282	-0.07842	-77.8269	-2.80828
26	Sig_NGMV	2.575	first_40	0.385019	19.388893	4.638127	-0.18437	10.47628	-0.08329	-76.3541	-2.8075
30	Sig_NGMV	3.291	first_40	0.322525	19.776949	4.45024	-0.14575	8.77856	-0.08482	-75.0738	-2.86364
34	Sig_NMS	1.96	first_40	-3.89046	148.35559	**3.286394**	-0.48803	-108.171	**-0.03971**	-100	-23.0445
38	Sig_NMS	2.326	first_40	-2.35723	147.88107	**3.259354**	-0.49077	-65.029	**-0.02946**	-100	-22.9555
42	Sig_NMS	2.575	first_40	-4.40932	147.5563	**3.243693**	-0.47727	-122.926	**-0.04344**	-100	-22.8772
46	Sig_NMS	3.291	first_40	-2.76716	148.51054	**3.768534**	-0.53043	-76.4971	**-0.0321**	-100	-23.0993
50	Sig_NMV	1.96	first_40	**0.988994**	21.298385	13.23671	-0.0048	**26.82994**	-0.04747	-78.5628	-3.04634
54	Sig_NMV	2.326	first_40	**1.179502**	21.245835	13.71045	0.039218	**31.96805**	-0.03862	-75.9694	-3.03151
58	Sig_NMV	2.575	first_40	**0.789036**	21.27864	13.92239	0.010878	**21.42653**	-0.05691	-79.1066	-3.04592
62	Sig_NMV	3.291	first_40	**0.858398**	21.283078	12.62834	0.04482	**23.30209**	-0.05364	-78.9206	-3.05746

	Strategy	**Theta**	**Number Asset**	**Mean Return**	**Std Return**	**Excess kurtosis**	**Skewness**	**Cumulative Return**	**Sharpe Ratio**	**Max Drowdown**	**CVaR 95%**
3	GMV	0	first_80	**0.152405**	17.783012	4.925048	-0.3051	**4.151713**	-0.1039	-65.8143	-2.55925
7	MV	0	first_80	-31.3629	125.11554	9.90855	-0.6753	-1025.2	-0.26666	-100	-20.2185
11	MS	0	first_80	-38.14476	155.95684	**2.236931**	-0.43108	-1308.34	-0.25741	-100	-23.8392
15	EW	0	first_80	-0.138636	22.855727	3.791284	-0.12184	-3.78209	**-0.09357**	-79.0812	-3.34615
19	Sig_NGMV	1.96	first_80	**0.336108**	18.998057	4.33823	-0.00949	**9.147632**	**-0.08758**	-70.5302	-2.71268
23	Sig_NGMV	2.326	first_80	**-0.839914**	19.02884	**4.327856**	0.007081	**-22.994**	**-0.14924**	-74.6823	-2.71993
27	Sig_NGMV	2.575	first_80	**-0.250929**	19.073195	**4.688922**	-0.02676	**-6.84938**	**-0.11802**	-72.6936	-2.72773
31	Sig_NGMV	3.291	first_80	**-0.139511**	19.057408	**4.794595**	-0.01407	**-3.80599**	**-0.11227**	-72.6678	-2.71404
35	Sig_NMS	1.96	first_80	-30.09637	204.56073	**4.32558**	-0.26808	-975.427	-0.1569	-100.006	-31.7076
39	Sig_NMS	2.326	first_80	-29.04927	204.06273	4.549666	-0.23737	-934.95	-0.15216	-100.005	-31.5478
43	Sig_NMS	2.575	first_80	-28.06335	207.86116	4.98697	-0.28713	-897.378	-0.14463	-100.007	-32.1794
47	Sig_NMS	3.291	first_80	-30.39896	206.714	4.908066	-0.27505	-987.236	-0.15673	-100.006	-32.062
51	Sig_NMV	1.96	first_80	-2.135433	29.554603	84.716	-0.73124	-58.844	-0.13993	-94.796	-4.02865
55	Sig_NMV	2.326	first_80	-2.11274	29.595249	85.04483	-0.64644	-58.212	-0.13897	-95.3761	-4.02663
59	Sig_NMV	2.575	first_80	-1.793517	29.618353	86.28899	-0.64515	-49.3367	-0.12808	-94.7642	-4.01578
63	Sig_NMV	3.291	first_80	-2.219995	29.593136	84.49106	-0.69751	-61.2005	-0.1426	-95.3695	-4.02691

Appendix B

Figure 15: Asset weights over time for the KFRP portfolio - 3 months rebalancing (Fama and French 5 factors)

Figure 16: Asset weights over time for the KFRP portfolio - 6 months rebalancing (Fama and French 5 factors)

The time-series plot of KFRP portfolio weights reveals notable shifts in portfolio allocation during specific periods of market stress: between 2003–2005, 2008–2010, and most significantly in 2014–2016. These spikes reflect the model's responsiveness to changing market regimes, likely driven by increased factor volatility or structural economic shifts, such as the global financial crisis or Japan’s economic stimulus policies under Prime Minister Shinzo Abe. The consistently positive and negative weights confirm a long-short structure, with the strategy aiming for balanced risk contributions rather than directional bets. Notably, weight volatility declines after 2017, suggesting greater model stability or more stable market conditions. Overall, the broad dispersion of weights implies strong diversification, though the large reallocations in turbulent periods may lead to higher transaction costs, highlighting the importance of understanding the model’s sensitivity and the drivers behind its shifts.

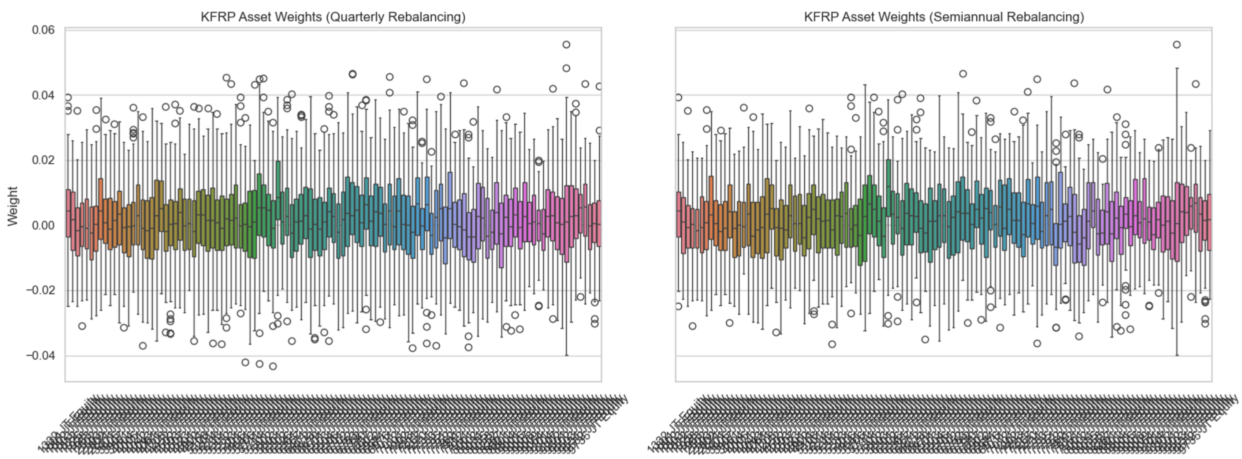

Figure 17: Asset weights over time for the KFRP portfolio - 3 - 6 months rebalancing (PCA)

Bibliography

Bhansali, V., Davis, J., Rennison, G., Hsu, J., & Li, F. (2012). The risk in risk parity: A factor-based analysis of asset-based risk parity. *The Journal of Investing, 21*(3), 102–110.

Borovkova, S., & Bosch, C. (2024). Allocating portfolio tail risk: Kurtosis-based factor risk parity. *SSRN*. https://ssrn.com/abstract=4943995 or http://dx.doi.org/10.2139/ssrn.4943995

Braga, M. D. (2015). *Risk-based approaches to asset allocation: Concepts and practical applications*. Springer.

Braga, M. D., Nava, C. R., & Zoia, M. G. (2023a). Kurtosis-based risk parity: Methodology and portfolio effects. *Quantitative Finance, 23*(3), 453–469.

Braga, M. D., Nava, C. R., & Zoia, M. G. (2023b). Kurtosis-based vs volatility-based asset allocation strategies: Do they share the same properties? A first empirical investigation. *Finance Research Letters, 54*, 103797.

Brockwell, P. J., & Davis, R. A. (2002). *Introduction to time series and forecasting*. Springer.

Chen, K.-T. (1958). Integration of paths—A faithful representation of paths by noncommutative formal power series. *Transactions of the American Mathematical Society, 89*(2), 395–407. https://doi.org/10.2307/1993193

Chevyrev, I., & Lyons, T. (2016). Characteristic functions of measures on geometric rough paths. *The Annals of Probability, 44*(6), 4049–4082.

Cong, L. W., Tang, K., Wang, J., & Zhang, Y. (2020). AlphaPortfolio for investment and economically interpretable AI. *SSRN Electronic Journal*. https://doi.org/10.2139/ssrn.3554486

Eklund, S. E. (2002). Empirical studies of neighborhood shapes in the massively parallel diffusion model. In *Lecture Notes in Computer Science* (pp. 185–194). https://doi.org/10.1007/3-540-36127-8_18

Flint, G., Hambly, B., & Lyons, T. (2016). Discretely sampled signals and the rough Hoff process. *Stochastic Processes and Their Applications, 126*(9), 2593–2614. https://doi.org/10.1016/j.spa.2016.02.011

Gregnanin, M., De Smedt, G. J., & Parton, M. G. (2024). Signature-based community detection for time series. In *Complex Networks & Their Applications XII* (Vol. 1142, pp. 146–158). Springer.

Gregnanin, M., Zhang, Y., De Smedt, J., et al. (2024). Signature-based portfolio allocation: A network approach. *Applied Network Science, 9*, 54. https://doi.org/10.1007/s41109-024-00651-1

Lemercier, M., Salvi, C., Damoulas, T., Bonilla, E., & Lyons, T. (2021). Distribution regression for sequential data. In A. Banerjee & K. Fukumizu (Eds.), *Proceedings of the 24th International Conference on Artificial Intelligence and Statistics* (Vol. 130, pp. 3754–3762). *Proceedings of Machine Learning Research*.

Liao, S., Ni, H., Szpruch, L., Wiese, M., Sabate-Vidales, M., & Xiao, B. (2023). Conditional Sig-Wasserstein GANs for time series generation. *arXiv*. https://arxiv.org/abs/2006.05421

Lyons, T. J. (1998). Differential equations driven by rough signals. *Revista Matemática Iberoamericana, 14*(2), 215–310.

Lyons, T. (2014). Rough paths, signatures and the modelling of functions on streams. *arXiv*. https://arxiv.org/abs/1405.4537

Martin, R. D., Stoyanov, S. V., Li, K., & Shammaa, M. (2023). Robust statistics for portfolio construction and analysis. *The Journal of Portfolio Management, 49*(9), 105–139.

Prigent, J.-L. (2007). *Portfolio optimization and performance analysis* (1st ed.). Chapman and Hall/CRC Financial Mathematics Series.

Sharpe, W. F. (1994). The Sharpe ratio. *The Journal of Portfolio Management, 21*(1), 49–58. https://doi.org/10.3905/jpm.1994.409501

Sutton, R. S., & Barto, A. G. (2005). *Reinforcement learning: An introduction*. IEEE Transactions on Neural Networks, *16*(1), 285–286. https://doi.org/10.1109/tnn.2004.842673

Vaswani, A., Shazeer, N., Parmar, N., Uszkoreit, J., Jones, L., Gomez, A. N., Kaiser, L., & Polosukhin, I. (2017). Attention is all you need. *arXiv*. https://arxiv.org/pdf/1706.03762v5

Vidović, J. (2020). Turnover-based illiquidity measurement as investment strategy on Zagreb Stock Exchange. *American Journal of Operations Research, 10*, 1–12. https://doi.org/10.4236/ajor.2020.101001