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# Optimal prize allocations in group contests

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# Introduction

Most political, economic and social activities are carried out by groups, not individuals. Such situations include lobbying, patent races and rent-seeking contests. In various competitions the reward, or part of it, has private characteristics and can be allocated in any way among winning group members.

The economics literature mostly focuses on such examples assuming symmetry within groups and an egalitarian prize distribution. The analysis of the effects of heterogeneity in private good contests has been mainly confined to differences in group size. The main result of this literature is the so-called "group size paradox": larger groups do not exert higher aggregate effort in equilibrium than smaller groups. There are two main reasons for this. First, the larger the group, the smaller the perceived effect of individual free-riding. Second, if the prize has private characteristics, the larger the group the smaller is the individual prize. As a result, the total group's effort is suboptimal.

Recently, Esteban and Ray (2001) have shown that this "common wisdom" does not necessarily hold. The authors, assuming symmetry among agents and an egalitarian prize allocation, derive sufficient conditions ensuring that the group size paradox is reversed. However, the assumption of symmetry among agents is very restrictive. Individuals in groups usually have different positions; politically, economically and culturally. More productive agents are typically rewarded with a higher share of the prize and tend to be more effective regarding the final outcome of the contest.

To the best of our knowledge, the work that investigates prize allocation among group members assumes symmetry among agents. As a consequence, two kinds of allocation have been suggested in the literature: the egalitarian rule and the relative effort rule. The first assigns the prize equally among group's members. The second assigns the prize according to the relative effort of members on total group effort. Under the latter, the free-riding within groups is eliminated because agents compete between and within groups' members. However, it can be applied only for a few specific cases; see Bandiera et al. (2011) for a specific example about fruit picking. We can list two main problems with this allocative rules. The egalitarian one is a consequence of the symmetry assumption among agents. There is no evidence that it is also the most effective in terms of exerted effort. Furthermore, under the relative effort rule, players do not know in advance what they get if they win. This is not suitable in any contest.

In this work, we explore an optimal allocative rule, assuming that at least a part of the reward can be divided among team members. Namely the prize is not a pure public good.

We seek answers to the following questions:

- given a contest between two groups whose members are heterogeneous in their ability, how should the prize be divided among players in order to maximize group probabilities of winning?
- Which rules govern this optimal choice?
- Do smaller groups outperform larger ones?

These questions are relevant in different situations. Organizers can choose a compensation scheme for the individuals composing each team, in order to maximize the overall effort exerted in the contest. For example, a retail firm can set up a contest in which it rewards the shop that achieves higher sales, it can also divide the prize differently according to each of its employees' responsibilities, offering a higher prize for sales managers and a lower one for sellers. Moreover, even if the organizer cannot decide how to allocate the prize within groups, we assume the presence of a team manager that has the only objective of maximize the probability of winning of its team by optimally allocating the prize among group members according to the difference in their ability. (Since this scenario leads to the same result, we consider only the latter one).

The work is organized as follows. Chapter 1 presents an overview of contest theory. Chapter 2 analyzes the basic model of rent-seeking and introduces a different approach to find the Nash equilibrium in this game. Chapter 3 focuses on the three main models regarding contests between groups. Finally, Chapter 4 presents our contribution as a model to study the optimal prize allocation. Moreover, we seek to shed light on the group size paradox, relaxing the common assumption of homogeneity among players.

# Chapter 1

# Contest

## 1.1 Overview

Nowadays there is a large interest and a growing literature on the theory and application of contests (Corners and Hartley 2001). An early contribution came from Tullock (1980) who studied a contest where two identical risk-neutral players compete to win a prize. Tullock, who was interested in the dissipation of a monopoly rent during a rent-seeking contest, employed a formulation of the contest success function in which the probability of winning for each player depends on the effort exerted by all contestants involved in the contest. The Tullock model has been extended in many directions; see Varian (1980), Baye et al. (1996). Examples include R&D, patent races, the periodic competitions between cities and countries to host events such as the Expo and competitions between universities to select most proficient students.

Afterwards, several authors devoted attention to contests between groups. Katz et al. (1990), Ursprung (1990), Baik (1993) used an extension of the Tullock model, while Baik and Lee (2001) applied an all-pay auction setting to analyze competition between groups. Recently, Baik (2008), Ryvkin (2011) and Ryvkin (2013) studied how the aggregate effort exerted in a contest depends on how players are sorted into groups.

## 1.2 Characteristics

At its simplest, a contest is a situation in which two or more agents compete exerting effort to win a prize. Many contests arise naturally, while others are designed by organizers in order to achieve specific goals. Regardless of how and why contests arise, in all of them it is possible to find the following three elements: contestants, prizes and effort.

- The contestant is the agent who takes part in a contest and can be an individual or a group. In contests between individuals, each player's choice is influenced by the opponents' behavior. Differently, in contests between groups, each player's choice depends on the behavior of both teammates and rivals.

- The prize is the reason why contestants exert effort. Each contest has its own type and number of prizes. Thus we can find contests where the prize is unique (auction) and others where multiple prizes are assigned (gold, silver, and bronze medals).

- Effort is the channel by which players can improve their probability of winning a contest. Two different environments are considered. The first is the one in which all players know the ability of all contestants. The second one occurs when agents are subject to adverse selection (the cost of effort is known only to themselves) and moral hazard (the outcome is observed, but the effort is unknown).

In practice, contests combine all these types of elements. Thus we are not able to classify contests in strictly different categories. However, we can analyze each element that composes a contest in order to have a better understanding of how it affects outcomes.

### 1.2.1 Contestants

Usually contests arise between individuals or between groups. This is a substantial difference in contest analysis since the behaviour of a player drastically changes if he is competing as part of a team or as a single player.

Consider a contest between two players. The probability of winning for player 1 depends on his own effort and on his rival's effort. Thus, in the simplest contest between two players, the probability that player 1 wins the competition is a function $p_1(x_1, x_2)$ and accordingly for player 2 is $1 - p_1(x_1, x_2)$.

In contests between groups, each team's probability of winning follows the same principle of competition between individuals. Thus, in a contest that involves two teams, the probability of winning for team $i$ is $p_i(X_i, X_j)$ and the probability of winning for team $j \neq i$ equals $1 - p_i(X_i, X_j)$. The effort exerted by a team can be considered as the sum of the effort exerted by all its members, $X_i = \sum x_{ik}$. Differently from individual competition, each player has a probability of winning which should be represented by a function such that $p_{i1}(x_{i1} + \sum_{k>2}^{n} x_{ik}, X_j)$. Here, given the efforts exerted by rival teams, a player $ik$ can improve his chance of winning in two different ways. First, by increasing his own effort; and secondly, through an increase in the effort of his teammates.

The most common problem related to team competition is free-riding, which occurs when players take advantage of the effort exerted by teammates and receive prize benefits without expending resources.

### 1.2.2 Effort

The effort is the instrument that allows players to increase their probability of winning the prize. It can be represented by different variables depending on where the contest takes place. For instance, in a contest between employees, the variable that better represents effort is labor time. In sports, the direct effort is that expended by team players, which is combined, developed, and coached by the team manager and the staff. This suggests, consistently with the case in which contestants are represented by groups, that the overall effort of a sport team can be better described by a production function that includes many inputs.

A common feature of many contests is that, independently of succeeding, players bear some costs. The representation of such costs in the contest analysis follows the same principles of the production function theory. In fact, we can define the cost of effort as $g(x_{ik}) = v_{ik}x_{ik}^{\alpha}$, where $g(.)$ is an increasing function and $v_{ik}$ the player's ability. In order to avoid the possibility that players exert infinite effort, we naturally posit $v_{ik} < \infty$.

Another source of analysis is the environment where contests take place. Indeed, the amount of effort does not depend only on the cost function faced by each player, but also on whether that cost of information is available to all contestants. We can have an environment with complete or incomplete information. In environments characterized by complete information, each player knows the abilities and the amount of effort exerted by all players. Alternatively, contests may be characterized by incomplete information. For instance, in job search applicants do not know the abilities of their competitors. The effort exerted may vary due to this lack of information.

Two main problems are related to incomplete information: the first is adverse selection, that may lead to undesired outcomes such as underprovision of effort due to players asymmetric information. The second is the moral hazard problem, that in team competitions is strictly related to free-riding. In this situation players are induced to take more risk (expending less effort), knowing that their behaviour is detrimental for their teammates. This occurs because members do not know the effort exerted by each player, and the only available information is the final outcome, which can by influenced by many other variables.

### 1.2.3 Prize

The prize is the award assigned to the winner for succeeding in the contest. In most cases it is well defined and known by players before the contest takes place. Other things being equal, it is reasonable to suppose that an increase in the prize value leads to an increase in the overall effort exerted by contestants. However, the amount of effort exerted does not depend only on the player's cost of effort function and the prize $P$, but also on the subjective value that each player assigns to the prize. Suppose that a prize $P$ is assigned to the winner. Each player maximizes his expected payoff, taking into account the prize value $P$ and his preferences. Then, the prize value for each player can be represented as $V_i = u_i P$, where the higher is the personal preference $u_i$, the higher is $V_i$.

Another element of analysis is the number of prizes that are assigned in a competition. This variable is relevant since each contestant has to maximize his own expected utility, taking into account the value and the number of prizes. Suppose that only one prize is awarded, which is the most studied case in the literature. This competition is known as the winner-takes-all contest, and it is designed to reward only the most efficient, or the most able player. In such situation each contestant perceives two different payoff alternatives. He either wins the prize or gets nothing.

Contests different from the winner-takes-all are carried out on a multi-prize distribution basis. For example, employees spend resources in order to be promoted in organizational hierarchies, which consist of different types of well-known positions; athletes compete for gold, silver, and bronze medals. Differently from the unique prize system, if there are at least three contestants and two prizes, each contestant foresees three possible payoffs: he can win the first prize, the second one, or nothing. In this case provision of effort are influenced by differences in expected payoff.

Least but not last, prizes may be a private, public, or not pure public good. If the prize is assigned in a contest between individuals, the type of prize does not matter (the winner takes it all). Instead, in a contest between groups, the public or private nature of the prize's characteristics is extremely important. If it is a public good, no team members can be excluded from its benefits. In the case of a private good, the prize can be shared among group members. Hence, some members, even if they are in the winning group, may be excluded from the prize allocation.

## 1.3 Contest success function

The contest function defines the probabilities of winning and losing via the effort exerted by contestants. The way in which the effort exerted by agents is translated in probabilities of winning and loosing is a crucial point in the analysis.

A general characteristic for several types of contests is that, given the rival effort, each player's probability of winning is increasing in its own effort. Consequently, each player's probability of winning is decreasing in the effort of the rival. Hence, the variables included in the contest success function may be the amount of effort exerted by each participant.

One of the functions[1] that satisfies these requisites is the Tullock success function defined as

$$p_i = \frac{x_i}{X}$$

There are two main reasons why the Tullock success function is widely applied in economics. First, Baye and Hoppe (2003) have identified conditions under which many contests are equivalent to the Tullock contest. Second, Skaperdas (1996) has provided axiomatic justifications for its application.

### 1.3.1 Axiomatizations

Tullock success function is similar to a probabilistic choice function used in other disciplines, see Luce and Suppes (1965). "A number of papers which investigated contests have employed the Tullock success function without any particular reason than analytical convenience (Skaperdas, 1996). However, a better understanding of limitations and advantages can be gathered using a few easily interpretable axioms.

[1]Another contest success function is in Hirshleifer (1989) where the probabilities of winning depend on the differences in effort.

Let $N = (1, 2, 3, .., n)$ be a set of players engaged in a contest. Denote the effort exerted by contestants with $x_i$, where $i \in N$. The Tullock success function returns the probability of winning for each player for any effort level. The following axiomatization improved by Skaperdas (1996) does not depend on the specification of the contest or on any particular level of effort.

**n-player contests**

Let $x = (x_1, x_2, ..., x_n)$ denote the vector of effort for the $n$ players involved in the contest. Denote each player's probability of winning with $p_i(x)$, where $p_i : [0, X]^n \to R$ and $X > x_i$ for all $i \in N$).

The following properties are maintained:

1) $\sum_{i \in N} p_i(x) = 1$ and $p_i(x) \geq 0$ for all $i \in N$ and all $x$; if $x_i > 0$, then $p_i(x) > 0$.

2) For all $i \in N$, $p_i(x)$ is increasing in $x_i$, and decreasing in $y_j$ for all $j \neq i$.

3) For any permutation $\pi$ of $N$ we have

$$p_{\pi(i)}(x) = p(x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(n)}$$

Property (1) says that the Tullock success function satisfies the condition of a probability distribution function. Moreover, if a player exerts a positive level of effort, the probability of winning cannot be zero. Property (2) says that a player's probability of success is increasing in its own effort, and decreasing in the effort of other players. Property (3) states that each player's probability of winning depends only on his own effort, and should not depend on player's identity. Moreover, it implies that if two players exert the same effort, their probability of winning must be equal, and if all players expend identical effort, the probability of succeeding must equal to $1/n$.

For any $k \in N$ let $x_{-k} = (x_1, ..., x_{k-1}, x_{k+1}, ..., x_n)$. Let $p_1, p_2, ..., p_n$ be functions that satisfy (1–3). Then, there is a function $p : [0, Y]^n \to R$ which is increasing in the first argument and decreasing in the other $n-1$ arguments; for any given vector of effort $x^0 = (x_1^0, x_2^0, .., x_n^0)$ we have

$$p_k(x^0) = p(x_k^0, x_{-k}^0); \forall k \in N$$

Given properties (1–3), it is guaranteed that for each player the probability of success is given by the same function.

Functions that satisfy properties (1–3) include:

i) $\frac{e^{kx_i}}{\sum_{j \in N} e^{kx_j}}$ where $k > 0$;

ii) $x_i^m - (\sum_{j \neq i} x_j^m)/(n-1) + \frac{1}{n}$ where $x_j^m \leq 1/n$ for all $j$;

iii) $\frac{x_i^m}{\sum_{j \in n} x_j^m}$ where $m > 0$.

Notice that the for $m = 1$ the function (iii) , corresponds to the Tullock Success Function.

**The CSF with fewer players**

Given that (i–iii) provide the player's probability of winning the contest, with every player moving independently from the others, what should be the most general contest success function for a subset of the original set of players? Moreover, if a nonempty set, $M \subseteq N$, engages a contest among the players in $M$, what is the probability of succeeding in that subset?

Let $p_{im}(x)$ be the winning probability of player $i$ who competes in the subset $M$. In addition, assume that $M$ contains at least two players. This can be assumed as follows:

4) $p_{im}(x) = \frac{p_i(x)}{\sum_{j \in M} p_j(x)} \forall i \in M$ and $\forall M \subseteq N$ that has at least 2 elements.

The property (4) implies that a contest among a lower number of players is quantitatively similar to one with a greater number of them. Take in consideration the function (ii), and suppose that two players are engaged in a contest. Applying (ii), the probability of winning for player 1 is

$$x_1 - x_2 + 1/2$$

However, adding a player (setting $n = 3$), we derive through (4), the probability of winning of player 1 if he participates in the contest against player 2 (with player 3 staying in the sidelines). That is,

$$\frac{x_1 - (1/2)x_2 - (1/2)x_3 + 1/3}{(1/2)x_1 + (1/2)x_2 - x_3 + 2/3}$$

which is different from the one above one.

This last equation shows the possible dependence of the outcome between a subset of players on the effort of players that do not participate in that specific contest. To avoid this possibility, the following property is applied.

5) $p_{im}(x)$ is independent on the effort of players not included in the subset $M$; or, $p_{im}(x)$ can be written as $p_{im}(x_m)$ where $x_m = (y_j; j \in M)$.

It follows that the outcome among players of $M$ does not depend on the effort of players that are not in $M$. The contest success function (ii) does not satisfy (5). However, function (i) and (iii), which is the Tullock success function, satisfy (4) and (5).

The application of contest success functions into a space lower than $n$ can be made not only in contest between a subset of players, but also in contests between groups. For instance, in a competition between three players, two of these can aggregate their effort against the third as an unique entity. Thus, if the two members in coalition expend an effort $x_1$ and $x_2$ against the player who exerts $x_3$, the probability of winning of the group is $p_G(x_1 + x_2, x_3)$ and for player 3 equals $p_3(x_3, x_1 + x_2) = 1 - p_c(x_1 + x_2, x_3)$.

The properties (1–4) are satisfied if and only if the contest success function satisfies:

6) $p_i(x) = f(x_i)/(\sum_{j\in N} f(x_j))$ for all $i \in N$ and $p_{im}(x) = f(x_i)/\sum_{j\in M} f(x_j))$ for all $i \in M(\subset N)$, where $f(.)$ is a positive increasing function of its argument.

It follows that $f(.)$ is unique up to a positive multiplicative transformation. Moreover, the class of functions that satisfies (6) respect also (1)-(5). [2]

Since the Tullock success function respects all the above properties, included the extension of coalition of effort, its application has an axiomatic justification in contests between groups. In the next chapter, we present the basic model of contest improved by Tullock.

[2]The proofs can be found in the original paper, Skaperdas (1996).

# Chapter 2

# Contests between individuals

In this chapter, we present the basic model of Tullock as an easy introduction to rent-seeking contests. Moreover, we illustrate the important work of Cornes and Hartley (2005) that provides a clever approach to derive the pure Nash equilibrium in rent-seeking game.

## 2.1 A contest between two individuals

Consider a contest between two agents where a unique prize $P$ is assigned. Players are labeled $i$ and $j$. Players $i$ and $j$ irreversibly and simultaneously exert an effort $x_i \geq 0$ and $x_j \geq 0$, respectively. Remember that, for any given combination of effort each agent has a probability of winning which corresponds to the probability of losing for its rival. Denote the probability of winning for the player $i$ as $p_i(x_i, x_j)$, and for the player $j$ as $p_j(x_j, x_i)$.

The Tullock contest success function describes player $i$'s probability of winning the contest as follows:

$$p_i(x_i, x_j) = \begin{cases} \frac{x_i}{x_i + x_j} & \text{if } x_i + x_j \neq 0 \\ \frac{1}{2} & \text{if } x_i = x_j = 0 \end{cases}$$

Conditionally on the probability of winning, the payoff function is a linear function of prize, own effort, and the effort of the rival. That is,

$$\pi_i = \begin{cases} P - g(x_i) & \text{with probability } p_i(x_i, x_j) \\ -g(x_i) & \text{with probability } 1 - p_i(x_i, x_j) \end{cases}$$

where $g(x_i)$ is the cost function.

$P$, $g(x_i)$, $g(x_j)$ and the contest success function $\frac{x_i}{x_i+x_j}$ are common knowledge for both opponents. Thus, under the assumption of complete information and risk-neutrality the expected payoff of player $i$ is:

$$\pi_i = \frac{x_i}{x_i + x_j}(P - g(x_i)) + \frac{x_j}{x_j + x_i} - g(x_i) \tag{2.1}$$

where $(x_i, x_j) \neq (0, 0)$. When $x_i = x_j = 0$ the expected payoff is $\pi_i = P/2$.

We can rewrite equation (2.1) as:

$$\pi_i = \frac{x_i}{x_i + x_j}P - g(x_i) \tag{2.2}$$

Player's best responses are derived by maximizing $\pi_i(x_i, x_j)$ and $\pi_j(x_j, x_i)$ respectively to $x_i$ and $x_j$. Differentiating expression (2.2) with respect to $x_i$ leads to the following first order condition:

$$\frac{\partial E(\pi_i(x_i, x_j)}{\partial x_i} = \frac{x_j}{(x_i + x_j)^2}P - g'(x_i) \tag{2.3}$$

The second order condition is:

$$\frac{\partial^2 \pi_i(x_i, x_j)}{\partial^2 x_i} = \frac{-2x_j}{(x_i + x_j)^3}P - g''(x_i) \tag{2.4}$$

Notice that (2.4) is concave as long as $x_j \geq 0$, which always holds. Therefore, the first order condition is sufficient to maximize player $i$'s payoff function.

Reformulating expression (2.3), we get the best response for the player $i$ taken as given $x_j$. That is,

$$x_i^* = -x_j + \sqrt{\frac{x_j P}{g'(x)}}. \tag{2.5}$$

Simultaneously solving the best response functions we obtain the unique equilibrium in which players $i$ and $j$ exert effort equal to:

$$x_i* = \frac{Pg'(x_j)}{(g'(x_i) + g'(x_j))^2} \tag{2.6}$$

$$x_j* = \frac{Pg'(x_i)}{(g'(x_j) + g'(x_i))^2} \tag{2.7}$$

The expected equilibrium payoff for the player $i$ is:

$$\pi_i = \frac{x_i^*}{x_i^* + x_j^*}(P) - g'(x_i^*) \tag{2.8}$$

In Tullock (1980) both players have the same valuation for prize $P$ and a linear cost function. Let, $g(x_i) = g(x_j) = x$.

The best response functions for player $i$ and $j$ are respectively,

$$x_i = -x_j + \sqrt{Px_j} \text{ and } x_j = -x_i + \sqrt{Px_i}.$$

The unique symmetric equilibrium is $x_i^* = x_j^* = \frac{P}{4}$.

This probability function has all the features that we discusses in the previous chapter. It is increasing in the effort of the agent $i$ at a decreasing rate. It is decreasing in the effort of agent $j$, and if both players expend zero effort the probability of winning is 1/2. In addition, it can be extended to contests involving more than two participants.

## 2.2 A contest between $n$ individuals

Suppose that there are $n$ risk-neutral players who compete for a prize $P$. The probability of winning for agent $i$ is

$$p_i(x_i, x_j) = \begin{cases} \frac{x_i}{x_i + \sum x_j} & \text{if } x_i + \sum x_j \neq 0 \\ 1/n & \text{if } x_i = \sum x_j = 0; \end{cases}$$

where $\sum x_j$ corresponds to the overall effort exerted by players $j \neq i$.

The expected payoff of player $i$ is

$$\pi_i = \frac{x_i}{x_i + \sum x_j} P - x_i \tag{2.9}$$

and the corresponding best response is given by

$$x_i^* = -\sum x_j + \sqrt{\sum x_j P}.$$

Due to the symmetry assumption we know that all players involved in the contests face the same best response function and therefore they exert the same level of effort, $x_i = x_j$. Thus, we have $\sum x_j = (n-1)x_i$ yielding the following unique symmetric equilibrium:

$$x_i^* = \frac{n-1}{n^2} P.$$

Replacing the unique effort equilibrium in the expected payoff we find that player $i$ exerts positive effort if and only if

$$\frac{n}{n-1} \geqslant P$$

Suppose that $P = 1$ and $n > 1$, it is straightforward that the constraint always holds.

The overall effort $X^*$ exerted in the contest is

$$X^* = \frac{n-1}{n} P.$$

In the following section we present the use of a theoretical quantity, hereafter called "share function", that was first applied by Cornes and Hartley in order to prove the existence of the Nash equilibrium in contests with convex cost of effort.

## 2.3 The share function

In this section we present the work of Cornes and Hartley that considers asymmetric contests between singletons. We present an alternative approach to solve the basic model of contest with linear cost function analyzed in the previous sections. Moreover, we consider a contest with strictly convex cost of expending resources in order to find the unique equilibrium through the use of the share function.

Consider $n \geq 2$ players that compete in a winner-takes-all contest. The probability of winning for the player $i$ is determined by the Tullock success function $p_i = x_i/X$. Notice that the probability of winning can be considered as a share of total effort exerted in the contest. Since this ratio plays a crucial role in the following analysis, let us denote it as

$$\sigma_i = \frac{x_i}{X}.$$

Assuming that contestants are risk-neutral, the expected payoff of player $i$ is

$$\pi = \frac{x_i}{X}P - g(x_i);$$

where $g(x_i)$ describes the cost function and $P$ the value of the prize. In order to introduce the share function, we assume that the cost of expending resources is represented by a linear function.

**Nash equilibrium with linear costs**

Let the expected payoff of player $i$ be

$$\pi_i = \frac{x_i}{X}P - v_i^{-1}x_i.$$

Notice that it is strictly concave in $x_i$. Then, the best response for any given effort exerted by opponents is

$$\frac{X_j}{X^2}v_iP = 1.$$

Rearranging, we get

$$x_i^* = \sqrt{v_iPX_j} - X_j. \tag{2.10}$$

In the previous sections, in order to solve this game we replaced the best response of all other opponents in the optimal response of player $i$. However, due to asymmetry among players, this approach can get quite messy. Here is the approach proposed by Cornes and Hartley (2005).

Rewrite the aggregate effort exerted by any other player $j \neq i$ as $X_j = X - x_i$. Replacing it in Equation (2.10) we obtain

$$x_i = \sqrt{v_i P(X - x_i)} - (X - x_i)$$

After some calculations, we finally have

$$x_i = X - \frac{X^2}{v_i P}.$$

Dividing both sides for $X$ we get what the authors called "share function" $s(X)$. That is,

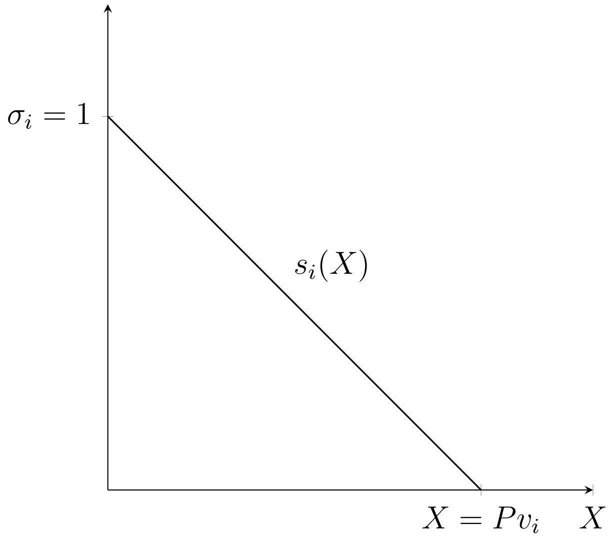
$$s(X) = \frac{x_i}{X} = 1 - \frac{X}{v_i P}.$$

**Proposition 2.3.1.** *If $g(x_i) = v_i x_i$ for contestant $i$, a share function exists for that contestant and satisfies*

$$s_i(X) = 1 - \frac{X^*}{v_i P}$$

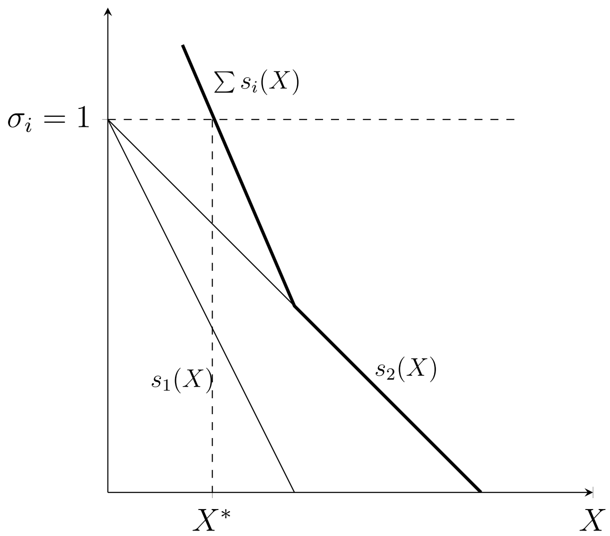
If contestant $i$ takes part in a contest in which $X^*$ is the total equilibrium outcome, the supply of effort is $x_i = s(X)X$. At that equilibrium, player $i$ probability of winning corresponds to the share $s_i(X)$.

The next figure represents the share function for risk neutral players with linear cost of effort.




Here, we can gather important properties of the share function useful for the subsequent analysis. It is continuous, its value tends to one as $X \to 0$, it decreases linearly[1] as $X$ increases until $X = Pv_i$ and it corresponds to zero thereafter.

The following figure provides the share function for two players with different cost parameters. Moreover, it shows the aggregate share function, obtained by summing up all $s_i(X)$.




[1]It decreases linearly because we assume linear costs of expending resources.

The unique Nash equilibrium occurs for $\sum \sigma_i = 1$. In a contest with linear cost of effort the unique equilibrium is given by

$$\sum s_i(X^*) = \sum 1 - \frac{X^*}{v_i P} = 1.$$

Moreover, given $X^*$, the amount of effort exerted by player $i$ at a given equilibrium point is $x_i^* = X^* s_i(X^*)$.

The properties of the individual share function imply the following for $\sum s_i(X^*)$:

i) it is continuous;

ii) for sufficiently small values of $X$ its exceeds one;

iii) for sufficiently large values of $X$ its value is zero;

iv) it is strictly decreasing in $X$ whenever it is positive.

There is no difference with respect to the classical problem analyzed in the previous section. In this paper, we have simply applied a different approach to derive the Nash equilibrium in pure strategy.

Let us apply this approach in a contest between $n$ players that compete for a given prize $P$.

Recall that the first order condition of the expected payoff is

$$\frac{X_j}{X^2}P - 1 = 0.$$

Rearranging and multiplying both sides for $x_i$ we get

$$x_i = v_i \frac{x_i X_j}{X^2} P;$$

where $x_i/X = \sigma_i$ is the share function, and $X_j/X$ is the probability of losing for players $i$, $X_j/X = (1 - \sigma_i)$. Assuming that all players are symmetric in their ability parameter $v_i$, we know that $\sigma_i = 1/n$. The unique optimal response is

$$x_i^* = \frac{n-1}{n^2},$$

and the aggregate effort is

$$X^* = \frac{n-1}{n}.$$

Notice that these results correspond to the unique equilibrium of the previous section.

## Nash equilibrium with convex costs

Throughout this section we maintain the following properties for the cost function:

**Assumption 1.1** For contestant $i$ the cost function $g(x_i)$ satisfies the following conditions: $g(0) = 0$, $g'(0) = 0$ and $g''(x) \geq 0$ for all $x > 0$.

Differently of the case of linear cost of effort, it is not possible to write down an explicit form for the share function. However, we can still apply the first order condition to obtain an implicit equation for $\sigma_i$ from which we can deduce qualitative properties.

Under Assumption (1.1) the first-order-condition is necessary and sufficient for the maximization. That is,

$$g'(x_i) = \frac{X_j}{X^2}P$$

Recalling that $\sigma_i = x_i/X$ the above equation can be rewritten as

$$g'(x_i)X = (1 - \sigma_i)P$$

**Proposition 2.3.2.** *If Assumption (1.1) holds for contestant i there exists a share function:* $s_i(X)$. $s_i(X)$ *satisfies* $s_i(X) = 0$ *if and only if* $g'(0)X > P$. *Otherwise,* $s_i(X) = \sigma$, *where* $\sigma$ *is the unique solution of:*

$$g'(\sigma X)X = (1 - \sigma)P. \tag{2.11}$$

The next proposition lists the properties of the share function for a convex cost of effort.

**Proposition 2.3.3.** *If Assumption (1.1) holds for contestant* $i$, *the share function* $s_i(X)$ *has the following properties:*

*ii)* $s_i(X)$ *is continuous;*

*ii)* $lim_{X \to 0} s_i(X) = 1$ *and* $lim_{X \to \infty} s_i(X) = 0$;

*iii)* $s_i(X)$ *is strictly decreasing where positive.*

Proposition (2.3.3) implies that contestant $i$ exerts a positive effort at any equilibrium. There is no value of $X$ at which the contestant does not exert effort.

Recall that $X^*$ is the aggregate effort in equilibrium if and only if the sum of the share function equals one at $X^*$. Since $s_i(X)$ is continuous and the aggregate share function is strictly decreasing, the equilibrium is unique. Moreover the uniqueness of $X^*$ implies a unique best response for the player $i$.

Let us apply this approach considering a contest among $n$ identical players with convex cost of effort equal to $x_i^\alpha$, where $\alpha > 1$. The maximization of the expected payoff is

$$\alpha x_i^{\alpha-1} = \frac{X_j}{X^2} P$$

Multiplying both side for $x_i$ we get

$$x_i = (\frac{(1 - \sigma_i)\sigma_i}{\alpha})^{\frac{1}{\alpha} P}.$$

Assuming that players are symmetric we can easily obtain the optimal response, that is

$$x_i^* = (\frac{n-1}{n^2\alpha})^{\frac{1}{\alpha}}P$$

The approach follows Cornes and Hartley (2005) and it is just an alternative way of presenting the same results. In Chapter 4, for an analytical convenience, we apply the share function in order to solve a contest between two heterogeneous groups.

# Chapter 3

# Contests between groups

In Chapter 2 we have analyzed contests among singletons with the aim of finding the game equilibrium. However, since we are considering winner-takes-all contests in competitions among individuals, players are indifferent between a public reward and a private one as they will not share the prize. Differently in group contests, the type of prize is extremely important both for the group effectiveness and for players' expected payoffs.

Consider a group contest for a public good. None of the players in the winning group can be excluded from the use of the prize. Thus, free-riders receive as much benefit as a player who expends resources. On the other hand, if a group increases its size, the individual's benefit remains unaltered. Basically, the problem related to public prizes is that free-riders can not be excluded from their use; yet, being the good public, the individual valuations of the prize do not decrease as group size increases.

If we differently consider the prize as a private good equally shared among members, an increase in group size causes a decrease of the individual rewards. Nevertheless, the private prize can be allocated in a way that excludes free-riders from its use.

So far we have considered contests among individuals with linear and convex costs of effort. The only difference between these two games is the total amount of effort exerted by players. In this chapter we investigate the importance of linearity and convexity of cost functions in group contests, presenting two models proposed by Baik (2008) and Ryvkin (2011). We eventually propose the model of Esteban and Ray (2001) that questions the group size paradox.

## 3.1 Public-good contests with linear costs

Consider a situation in which the government disposes of a budget for building a hospital, and several communities compete to win that budget. The government decides the winning group, based on the voluntary contribution made by each community. In other words, it is selected according to the Tullock success function. Moreover, all members in the winning community benefit from the construction of the hospital. That is, the prize is a pure public good.

Baik (2008) analyzes this kind of contest in order to investigate how strong is the free-riding in such a game. As we are going to see, only the player with the lowest cost exerts positive effort in this specific framework. All the other group members free-ride.

Consider two competing groups composed of $n_i$ players that choose simultaneously and independently their effort $x_{ik} \geq 0$. The group total effort is $X_i = \sum x_{ik}$. Agents in the winning group are rewarded with a public prize $P = 1$. The losing group receives zero. Recall that the group probability of winning is defined by the Tullock success function $X_i/X$, where $X = \sum X_i$. Players' costs of expending resources are $v_{ik}^{-1} x_{ik}$, where $0 < v_{ik} < \infty$ is a heterogeneous parameter, and $x_{ik}$ is the linear cost function. Function and parameters are common knowledge for all competitors.

For explanatory convenience, let us assume that ability parameters are ordered as follows.

**Assumption 2.1** *Without loss of generality, assume that:* $v_{i1} > v_{i2} > ... > v_{im} > 0$.

Player $ik$'s expected payoff is

$$\pi_{ik} = \frac{X_i}{X} - v_{ik}^{-1} x_{ik}. \tag{3.1}$$

Notice that Equation (3.1) is strictly concave with respect to $x_{ik}$. Clearly, this occurs because we are employing the Tullock success function. Indeed, given the opponent's contribution $X_j$, group $i$'s probability of winning is increasing in its own effort at a decreasing level and it is decreasing in the effort of the rivals. It follows that the first order condition is necessary and sufficient for the maximization.

Player $ik$'s best response is

$$\tfrac{X_j}{X^2} - v_{ik}^{-1} = 0 \text{ for } x_{ik}^* > 0 \text{ or } \tfrac{X_j}{X^2} - v_{ik}^{-1} \leq 0 \text{ for } x_{ik}^* = 0.$$

Clearly, if $x_{ik}^* > 0$, the marginal payoff $\frac{X_j}{X^2}$ equals the marginal costs. On the contrary, provided that $x_{ik}^* = 0$, the marginal payoff is lower than the marginal cost, and the optimal contribution is zero.

As a preliminary step to obtain the pure-strategy Nash equilibrium, it is necessary to obtain group $i$'s specific strategy. Thus, given the other groups' contributions $X_j^*$, the group $i$'s equilibrium strategy profile is identified as a vector of effort. Group $i$'s best response, calculated through player $ik$, is defined as the contribution that maximizes:

$$\frac{X_i}{X} - v_{ik}^{-1} X_i \tag{3.2}$$

subject to $X_i > 0$.

Equation (3.2) represents the best response of group $i$ taking into account the cost of contributing of player $ik$. In other words, it is the best response of the group $i$ as a whole, computed with player $k$'s cost of expending resources. The best response satisfies the first-order condition:

$$v_{ik}\frac{X_j^*}{X^2} - 1 = 0 \text{ for } X_i^* > 0 \text{ or } v_{ik}\frac{X_j^*}{X^2} - 1 \leq 0 \text{ for } X_i^* = 0.$$

Notice that the payoff function is strictly concave in $X_i$. It follows that $X_i^*$ is unique. Moreover, group $i$'s best response computed with a lower cost of expending resources is higher than the one computed with a higher cost.

**Proposition 3.1.1.** *Under Assumption (2.1) we have that* $X_i^*(X_j^*, v_{i1}) > X_i^*(X_j^*, v_{i2}) > ... > X_i^*(X_j^*, v_{ik})$.

Proposition (3.1.1) comes from a simple fact. The group $i$'s optimal contribution computed with the lowest cost of expending resources is greater than that computed with higher cost parameter. Now we are prepared to show the main result of this work.

Let the vector $\mathbf{v}$=$(x_{i1}^*, x_{i2}^*, .., x_{im}^*)$ represent group $i$'s best response given the other group's level of contribution $X_j^*$. At group $i$'s equilibrium, the optimal choice $x_{ik}$ is given by the maximization of the expected payoff.

Lemma 1 shows that the group effort level $X_i^*$ must be equal to player $i1$'s best response.

**Lemma 1** *Group $i$ effort level for any $X_j$ is equal to group $i$'s player $i1$ best response.*

Consequently, the total group $i$'s contribution is exactly the optimal response of the player with the lowest cost, $x_{i1}^*$. Then it is possible to construct the total contribution of group $i$ given $X_j^*$.

**Proposition 3.1.2.** *Let $X_i^* > 0$ and $v_{i1} > v_{i2} > ... > v_{im}$, there is a group $i$'s unique specific equilibrium at which $x_{i1} = X_i^*$ and $x_{il}^* = 0$ with $l > k$.*

Each group's total contribution exactly corresponds to player $i1$'s best response, taking as given $X_j$. Any other player whose cost of expending resources is higher than the cost of at least a player in his group, expends zero effort. Namely, he is a free-rider. Another implication is that the total effort exerted is equivalent to the effort obtained in a contest among $n$ singletons. Obtaining the effort level of the group is sufficient to solve a game among singletons that compete for a public prize. Moreover, the group effectiveness depends only on the player with the lowest cost of expending effort.

The interpretation of this result is straightforward. The player with the lowest cost of contributing has the lowest constant marginal cost with respect to all the other group's members, while the marginal payoff is the same for all the group's members. The optimal contribution $x_{ik}^*$ of the player with the lowest cost is given by the intersection between marginal revenue and marginal cost. It follows that, any other player $ik \neq i1$ does not contribute since their marginal costs are above the marginal revenue.
It is important to underline that this result depends on the linearity of the cost function. However, for a few specific cases the contribution of resources can be represented by a linear function. In many contexts the costs of expending resources are more likely to be strictly convex.

The next model, proposed by Ryvkin (2011), assumes a strictly convex cost of expending resources. As we see in the next section, through this assumption, the equilibrium of the game and of group's contributions drastically changes.

## 3.2 Public-good contests with convex costs

Ryvkin (2011) provides a model for optimally sorting players between teams in order to manipulate the total effort exerted in the contest. The Baik's model is not applicable in this situation. Indeed, there is no reason for sorting players if only one is active in every group. Thus, a model that predicts positive efforts for all players is more suitable for an analysis of sorting.

The author considers a model of contest between groups of heterogeneous players with a strictly convex cost of effort. In order to focus on the effect of sorting heterogeneous players, he considers the prize as a public good. In that way he avoids the effect of the decrease in individual reward due to private nature of the prize.

Consider $N$ risk-neutral players sorted in $m \geq 2$ groups of $n \geq 1$ players each. Groups are indexed by $i = 1, ..., n$ and players within groups by $ik$. As in Baik (2008), all players simultaneously and independently exert an effort $x_{ik} \geq 0$. The group $i$'s aggregate effort is defined as $X_i = \sum x_{ik}$. Given the public good characteristics of the prize, each player in the winning group receive a prize $P$ normalized to one. The probability of winning is given by the Tullock success function $X_i/X$. Moreover, players $ik$'s cost of expending resources is $v_{ik}^{-1} g(x_{ik})$. Functions and parameter are common knowledge for all players involved in the contest.

Players $ik$'s expected payoff is:

$$\pi_{ik} = \frac{X_i}{X} - v_{ik}^{-1} g(x_{ik}).$$

Notice that, if $g(x_{ik}) = x_{ik}$ this problem corresponds exactly to the analysis of Baik (2008). Here, the author assumes that the cost functions are strictly convex. Moreover, under additional assumption, it is possible to show the uniqueness of the equilibrium in which all players exert positive effort.

**Assumption 3.2**

i) $g(0) = 0$;

ii) $g'(0) = 0$;

iii) $g'(x) > 0$;

iv) $g''(x) > 0$;

v) $g'''(x)$ exists and it is finite.

Part (i) states that players do not bear costs when they do not exert any effort. Part (ii) states that the marginal cost at $x = 0$ are zero. Part (iii) and (iv) states respectively that the cost function is strictly increasing and strictly convex and ensure the existence and uniqueness of an equilibrium in which all players exert positive effort. Assumption (v) is a condition that is required for the development of the quadratic approximation used by the author in order to provide the optimal sorting.

Player $ik$'s best response to all other players' choice of effort is given by the first-order conditions. That is,

$$\frac{X_j}{X^2} = v_{ik}^{-1} g'(x_{ik}) \tag{3.3}$$

It is straightforward to show that there is a pure strategy Nash equilibrium in which all players exert positive effort. Notice that, if at least a players in group $j \neq i$ exert positive effort, the left-hand side of the Equation 3.3 is positive, that in turn implies that the right hand-side is also positive. It follows that the best response of all players is positive.

**Proposition 3.2.1.** *Under Assumption (2.1), the contest has a unique pure Nash equilibrium. The equilibrium level of effort $x_{ik}^*$ is positive and satisfies the system of Equation (*3.3*) with equality.*

*Proof.* Notice in Equation (3.3) for group $i$ the left hand side of the equation is the same for all $ik$. Therefore $v_{ik}^{-1} g'(x_{ik}) = v_{il}^{-1} g'(x_{il})$ for all $k, l$. Under Assumption (2.1) $g(.)$ is strictly convex, and then $g'(.)$ is strictly increasing. It follows that for a given $x_{ik}^*$, the equilibrium effort level of all other group members $ik$, for $k > 1$, can be uniquely determined as

$$x_{ik} = (g')^{-1} \frac{v_{ik}}{v_{i1}} g(x_{i1}).$$

The group $i$'s total effort is

$$X_i = x_{i1} + \sum_{k>1}(g')^{-1}(\frac{v_{ik}}{v_{i1}})g(x_{i1}).$$

Knowing that in equilibrium all player exert positive effort, and the only difference between group members is their ability parameter, to obtain the aggregate group $i$ effort it is possible to consider all other players optimal responses as a share of the $ik$. Thus, the system of $nm$ equations reduces to a system of $n$ equations for $x_{i1}$, equivalent to the contest among $n$ groups to a contest among $n$ individuals.

□

The main goal of this work is to explore how the aggregate group effort exerted in equilibrium depends on the sorting of players among groups. The approach followed can be summarized as follows. Assume first that players are homogeneous in their ability parameter $v_{ik}$. Clearly, all players involved in the contest exert the same effort $x^*_{ik}$. Consider now that players are heterogeneous with respect to $v_{ik}$ and that the average ability parameter is $v$. It follows that each new equilibrium effort can be identified through the deviation from the symmetric aggregate effort. If this deviation is small (from a mathematical perspective $d \sim 0.1$), it can be identified via the quadratic approximation.

**Proposition 2 Ryvkin (2011)** *The optimal sorting of players is the one that maximizes (or minimizes, depending on the cost function) the sample variance in aggregate ability across groups.*

The usefulness of the result of this work is its application for any type of cost functions. Indeed, for different cost function the optimal sorting may be the one that maximizes or minimizes the variance in ability across groups.

However, as claimed by Ryvkin, the quadratic approximation can provide a simple and clear result for sorting players, but incurs in some limitations. First, it is not precise and imposes restriction on the degree of heterogeneity. Second, this benchmark for sorting players is not applicable if groups are heterogeneous in their size. Thus, assuming that groups are fixed, but heterogeneous in their size, this result loses his universality. Indeed, it is applicable only if groups are composed of the same number of players.

In Chapter 4 we relax this assumption, considering a contest for a private good prize, allowing both heterogeneity within and between groups.

Differently from Baik (2008), the assumption of strictly convex cost of effort leads to all players involved in the contest to exert positive effort. Thus, the effect of free-riding is less acute. However, even if this work allows heterogeneity among players, it considers only a pure public good contest.

## 3.3 Group size paradox

The paper most related to our work is Esteban and Ray (2001). The authors are among the first to question the group size paradox. Olson (1965) argue that the free-riding problem makes smaller group more effective than larger ones and claims that larger groups are less effective in pursuing their goals. However, consider the government activities that transfer money on the basis of individual characteristics. We can include pensions, education, health benefits for which there is a fixed budget. For such examples, we should observe people organized in small groups in order to prevent the free-riding problem. We should see small organized lobbies that compete to become the sole beneficiaries of the public found. The true is that we do not see anything like this. What we see in the real world is that firms merge, labor unions bargain with the government has a unique entity, and political parties form alliances during elections. Esteban and Ray (2001) write:

> *The key question is the aggregate potency of the group, which is what determines effectiveness in the sense of success probabilities. Decreasing in personal contributions are not necessarily incompatible with increase in aggregate effectiveness.*

**Impure public good contests**

Consider a contest in which $N$ agents sorted in $n$ groups compete for a prize $P$. The prize can have mixed public-private characteristics. Thus, it is an impure public good.
Let the expected payoff be

$$\pi_{ik} = w - g(x_{ik})$$

where $g(x_{ik})$ is an increasing convex function, and $w$ is the per-capita benefit. In other words, from the benefit $w$ it is subtracted the cost of contributed effort. The marginal rate of substitution between reward and effort, the extra benefit that compensates individuals for contributing an extra unit of effort, increases as total effort increase due to the strictly convexity assumption on $g(x_{ik})$. Moreover, the MRS can be written as

$$MRS = g'(x)$$

and its elasticity, $\epsilon$, at any effort level as

$$\epsilon = \frac{xg''(x)}{g'(x)}.$$

$\epsilon$ is the key variable that determines the main result of this paper.

In this model the authors assume that the prize is an impure public good. Moreover, any private part of it is equally shared among group members (we question such point in Chapter 4). Define the per-capita benefit of each player $ik$ at a given equilibrium point as

$$w_i = \lambda P + (1 - \lambda)\frac{M}{n_i}.$$

Notice that for $\lambda = 0$ the prize corresponds to a pure private good, and for $\lambda = 1$ corresponds to a pure public good. Thus, $\lambda$ is a parameter that establishes the degree of privateness (publicness) of the prize.

Moreover, the perceived effect of publicness on $P$ depends on the group size. Thus, the larger the group, the higher is the perceived effect of the publicness. For any given $\lambda$, $P$ and $M$ the public part of the prize is

$$\theta_i(\lambda, n_i) = \frac{\lambda P}{\lambda P + (1-\lambda)(M/n_i)}$$

Let $X_i = \sum x_{ik}$ be the total effort of group $i$. The probability of winning for group $i$ is given by the Tullock success function. That is

$$p_i = (\frac{X_i}{X}).$$

Therefore, the expected payoff for player $ik$ is given by

$$\pi_{ik} = \frac{X_i}{X} w(\lambda, n_i) - g(x_{ik}).$$

The unique equilibrium is characterized by the first order conditions. That is,

$$[\frac{1}{X} - \frac{X_i}{X^2}]w(\lambda, n_i) - g'(x_{ik}) = \frac{1}{X}(1-p_i)w(\lambda, n_i) - g'(x_{ik}) = 0 \tag{3.4}$$

Each player $ik$ takes as given the effort exerted by everyone else, and chooses his $x^*_{ik}$ in order to maximize his expected payoff. [1]

We now provide the main proposition of this work that demonstrates for which conditions the group size paradox is reversed.

*Proof.* Consider the equilibrium of the game. Whenever $\epsilon > 1$, the probability of winning of group $i$ is strictly increasing in group size for all $\lambda \in [0,1]$. Irrespective on the degree of publicness (privateness) of the prize larger group outperform smaller ones. Moreover, the winning probabilities are increasing over a pair of group size $n$ and $n'$, where $n < n'$, if $\theta(\lambda, n_i) \geq 1 - \epsilon$.

□

[1] We omit the proof because the one provided in Ryvkin (2011) still holds.

The authors demonstrate this proposition as follows. They examine the behaviour of $p_i = X_i/X$, keeping $X_i$ unchanged at its equilibrium value. Moreover, they pretend that $n_i$ is a continuous variable (clearly, it is an integer) in Equation (3.4) in order to differentiate $p_i$ with respect to $n_i$. The final equation is:

$$\frac{\partial p_i}{\partial n_i} = \frac{p_i}{n_i}\frac{\epsilon - (1-\theta(\lambda, n_i))}{\epsilon + p_i/(p_i)}$$

Notice that the first order condition is positive for $\epsilon > 1$. However, proofs of such solutions are not provided in Esteban and Ray (2001). Thus, we constructed a proof by ourselves[2]

*Proof.* We can rewrite equation (3.4) as

$$\frac{1}{X}(1-p_i)[\lambda P + (1-\lambda)\frac{M}{n_i}] - g'(X\frac{p_i}{n_i})$$

where $x_{ik} = X\frac{p_i}{n_i}$, $w = \lambda P + (1-\lambda)\frac{M}{n_i}$,$\theta = \lambda P/w$ and $\epsilon = x_{ik}g''(x_{ik})/g'(x_{ik})$.

The authors assume that in Equation (3.4) $n_i$ is a continuous variable in order to differentiate $p_i$ with respect to $n_i$. That is,

$$\partial p_i(\frac{1}{X}w) + g''(X\frac{p_i}{n_i})\frac{X}{n_i} = \partial n_i(\frac{-(1-p_i)(1-\lambda)M}{X}\frac{1}{n_i^2} + g''(X\frac{p_i}{n_i})\frac{Xp_i}{n_i^2})$$

$$\partial p_i(\frac{wn_i + g''(x_{ik})X^2}{Xn_i}) = \partial n_i(\frac{-(1-p_i)(1-\lambda)M + g''(x_{ik})X^2p_i}{Xn_i^2})$$

After some tedious calculation we are able to provide the solution.

$$\frac{\partial p_i}{\partial n_i} = \frac{(1-p_i)[-(1-\lambda)M + \frac{g''(x_{ik})x_{ik}}{g'(x_{ik})\frac{g'(x_{ik})}{x_{ik}}}X^2\frac{p_i}{1-p_i}}{wn_i + \frac{g''(x_{ik})}{g'(x_{ik})}x_{ik}\frac{g'(x_{ik})}{x_{ik}}X^2}\frac{Xn_i}{Xn_i^2}$$

Reformulating and replacing $\epsilon = x_{ik}g''(x_{ik})/g'(x_{ik})$, we get

$$\frac{\partial p_i}{\partial n_i} = \frac{(1-p_i)[-wn_i + \lambda Pn_i + \epsilon\frac{Xp_iw}{x_{ik}}]}{wn_i + \epsilon X(1-p_iw)/x_{ik}}\frac{1}{n_i}$$

Recall that $\theta = \lambda P/w$. Rearranging the numerator we have

[2]I am grateful to Professor LiCalzi for his huge help with this solution.

$$\frac{\partial p_i}{\partial n_i} = \frac{1}{n_i} \frac{(1 - p_i)[-(1 - \theta - \epsilon)]}{1 + \epsilon \frac{p_i}{1 - p_i}}$$

Lastly, we get the result provided by Estaban and Ray. That is,

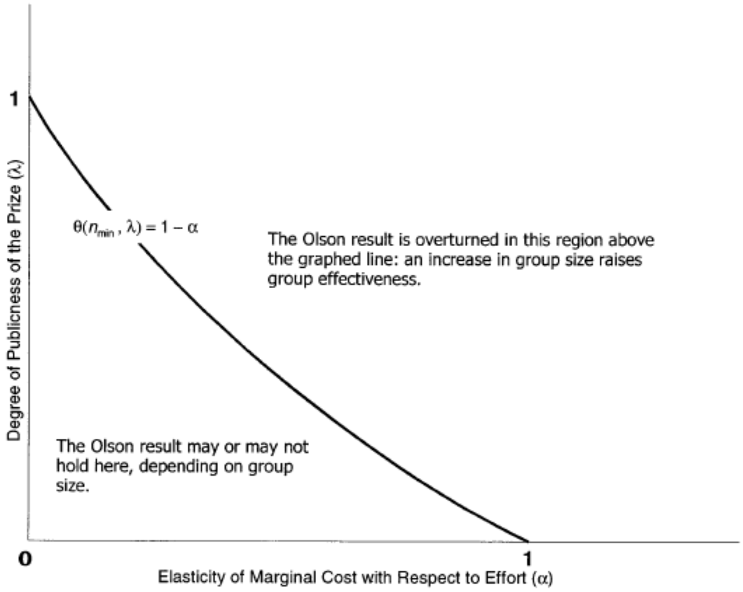
$$\frac{\partial p_i}{\partial n_i} = \frac{p_i}{n_i} \frac{\epsilon + \theta - 1}{\epsilon + \frac{p_i}{1 - p_i}}$$

□

Suppose now that the size of a group may increase, while all other groups sizes are fixed. Under the new equilibrium, the winning probability of group $i$ increases if

$$\theta(\lambda, n) \geq 1 - \epsilon.$$

This condition is satisfied for enough large groups, or when $\lambda$ is closed to 1 (the prize is a pure public good). Moreover, when $\epsilon > 1$, the probability of winning always increases irrespective of the value of $\lambda$. Assume now that the elasticity of marginal costs is constant, then $g(x_{ik}) = x_{ik}^{\alpha+1}$. It follows that also the elasticity of marginal cost is constant and equal to $\epsilon = \alpha$. Then we can provide the graphically the region from which the group size paradox is overturned.




Esteban and Ray (2011) provides a model that demonstrates, even if the prize is a pure private good, that the group size paradox is reversed for a sufficiently high cost steepness ($\alpha + 1 > 2$). Instead, for lower cost steepness the result is uncertain, and depends on the degree of publicness of the prize. However, their model did not consider the effect that may have heterogeneity among players. In Chapter 4 we include heterogeneity within groups. Moreover, under additional assumptions, we show that even if the cost of effort is not too steep and the prize is a pure private good, larger groups have at least an equal probability of winning than smaller ones.

# Chapter 4

# A model of contest

We present a game theoretic framework to investigate the optimal prize allocation among players in a winner-takes-all contest. In order to allow prize allocation, we assume that the prize assigned to the winner group has private characteristics; e.g., thus it may correspond to a monetary reward. In this framework we define the prize allocation as optimal if it maximizes the group probability of winning. As a consequence, no particular attention is given to the players' expected payoffs[1]. This is a strong restriction since group members could bargain among themselves over the prize distribution before the contest takes place. Hence, in order to avoid internal bargaining and keeping the model credible, we assume the presence of a team manager whose objective is to make the group win the contest.

We analyze two models of contest borrowed from Esteban and Ray (1999, 2001), and extend them in order to optimally distribute the prize among group members. We first study a common noncooperative model characterized by free-riding within groups. In this environment players take decisions independently. In other words, they care only about their favorite option.
Secondly, we investigate a cooperative model where players maximize group expected utility. In this framework, free-riding within groups is eliminated.

[1]The phrase "no particular attention is given to the players' expected payoffs" is ambiguous. The interpretation is that the manager does not matter if the prize allocation is not fair. For instance even if two players are identical, the manager may allocate the whole prize to a single player.

At the end of this chapter we want to provide answers to our initial questions:

- given a contest between two groups whose members are heterogeneous in their ability, how should the prize be divided among players in order to maximize the probability of winning?
- Which rules govern this optimal choice?

We seek answers to our questions in a specific framework. We consider $N = 4$ risk-neutral players sorted in two equal groups. Competing groups are indexed by $i = a, b$. Players within groups are indexed by $ik = i1, i2$. Groups compete in a contest structured as follows: all players irreversibly and simultaneously exert an effort $x_{ik} \geq 0$, and the group total effort is $X_i = x_{i1} + x_{i2}$.

The group that wins the contest is rewarded with a prize normalized to one. The group that loses receives zero. The prize is a pure private good and it may be allocated in any way among the winning players. We assume that players, before the contest takes place, are aware of the share of the prize that they receive if they win the contest. This share, defined with $\varphi_{ik}$, is established by a team manager, where $\varphi_{i1} + \varphi_{i2} = 1$. The group probability of winning is defined by the Tullock success function $\frac{X_i}{X}$. Defining $X$ for the aggregate effort, $X = \sum X_i$, the probability of group $i$ winning the contest can be thought as a share of the total effort exerted. This ratio, denoted by $\sigma_i$, plays a key role in subsequent analysis. That is, let

$$\sigma_i = \frac{X_i}{X}$$

Players' costs of effort are $v_{ik}^{-1} g(x_{ik})$; $0 < v_{ik} < \infty$ is a heterogeneous ability parameter and $g(x_{ik})$ is an increasing function. Functions and parameters are common knowledge for all agents, team managers included.

Conditional on the probability of winning, player $ik$'s expected payoff is:

$$\pi_{ik} = \frac{X_i}{X}\varphi_{ik} - \frac{g(x_{ik})}{v_{ik}}$$

We impose the following assumption on $g(x)$:

**Assumption 1**

i) $g(0) = 0$;

ii) $g'(0) = 0$;

iii) $g'(x) > 0$ for all $x > 0$;

iv) $g''(x) > 0$ for all $x > 0$.

Part (i) states that players do not bear costs when they do not exert any effort. Part (ii) states that the marginal cost of effort at $x = 0$ is zero. Part (iii) and (iv), state respectively that the effort cost function is strictly increasing and strictly convex. Finally part (iv), in conjunction with part (iii), ensures the existence and uniqueness of an equilibrium in which all players exert positive effort if they receive a strictly positive part of the prize. Assumption 1 is held throughout the paper.

We are aware that providing explicit cost functions is a very strong assumption. Nevertheless, to make the model tractable, we assume that $g(x_{ik}) = x_{ik}^{\alpha}$. Clearly, we postulate $\alpha > 1$ to satisfy Assumption 1.

Under Assumption 1, in order to optimally allocate the prize the team manager has to take into account:

- the possible heterogeneous ability parameter $v_{ik}$;
- the function $g(.)$;
- the cooperation or lack thereof among players.

## 4.1 Noncooperative model

In a noncooperative contest agents decide how much effort to exert simultaneously and noncooperatively. Agents' contributions are aggregated in every group, and the winning group is determined on the basis of overall group effort. Each player $ik$'s best response to all other players' choices of effort is given by the first order condition associated with the maximization of $\pi_{ik}$ as a function of $x_{ik}$, subject to $x_{ik} \geq 0$.

Player $ik$'s expected payoff is:

$$\pi_i = \frac{X_i}{X}\varphi_{ik} - v_{ik}^{-1} g(x_{ik})$$

Under Assumption 1, given the amount of other players' choices of effort, this function is strictly concave with respect to $x_{ik}$. The first-order condition is necessary and sufficient for the best response. If groups are composed of two agents, player $i1$'s best response is

$$\frac{X_{j \neq i}}{X^2}\varphi_i = v_{i1}^{-1} g'(x_{i1}); \tag{4.1}$$

and player $i2$'s best response is

$$\frac{X_{j \neq i}}{X^2}(1 - \varphi_i) = v_{i2}^{-1} g'(x_{i2}). \tag{4.2}$$

It is possible to show that there is a pure strategy Nash equilibrium in which groups exert positive effort. Assume that both team managers allocate the whole prize to a single player, for instance to player $i1$. Then $\varphi_i = 1$. This implies that player $i2$ faces the left-hand side of Equation (4.2) equal to zero and therefore his choice of effort is $x_{i2} = 0$. Otherwise for player $i1$, if any other player $j \neq i$ exerts a positive effort, the left-hand side of Equation (4.1) is positive. This implies that the right-hand side should also be positive. This, in turn, implies that the best response is positive also for player $j1$.

Suppose now that all players receive a positive share of the prize, whereby $0 < \varphi_i < 1$.

It follows that the left-hand side is positive for all players involved in the contest. The argument is completed by the fact that all players exerting zero effort is not an equilibrium. Indeed, under the assumption $g'(0) = 0$ each player exerts positive effort, except for $\varphi_i = 0, 1$. Players may not contribute only if $g'(0) > 0$. Otherwise, a strictly positive $x_{ik} > 0$ that satisfies the first-order condition always exists. Moreover, in this chapter we show that the equilibrium is unique.

We are interested in studying the effect of the prize allocation on probability of winning. As we have seen, allocating the whole prize to a single player leads the other group member to exert zero effort. Thus we begin our analysis with a simpler model, which allows us to investigate the effect of an increase or decrease of group members on the probability of winning. Moreover, we introduce a theoretical quantity called share function proposed by Cornes and Hartley (2005).

## 4.2 Group size effects in noncooperative contests

In the economic literature there is a general agreement that, due to free-riding problem, individuals tend to contribute a lower effort when the group to which they belong is larger. There are two main reasons for this. The larger the group, the smaller the perceived effect of individual free-riding. Second, if the prize has private characteristics, the larger the group the smaller is the individual prize. This is well known as the group size paradox. However, as pointed out by Esteban and Ray (2001), a decrease in personal contribution is not necessarily incompatible with an increase in aggregate effectiveness. Indeed, under some conditions the group size paradox is reversed.

The following analysis leads to the same result as Esteban and Ray (2001), but follows a simpler approach that directly calculates the group probability of winning.

Consider two competing groups composed of $n_i$ players. All players are symmetric in their ability parameter, $v_{ik} = \alpha^{-1}$. Moreover, agents in the winner group are rewarded with an equal share of the prize, $\varphi_i = 1/n_i$. Due to symmetry among players, we can rewrite the group total effort as $X_i = n_i x_i$. Recall that the group probability of winning is defined by the Tullock success function. Let $\sigma_i = \frac{X_i}{X}$.

The expected utility for player $ik$ is

$$\pi_{ik} = \frac{X_i}{X}\frac{1}{n_i} - \alpha^{-1}x_{ik}^{\alpha},$$

and his respective best response is

$$x_i = (\frac{X_j}{X^2}\frac{1}{n_i})^{\frac{1}{\alpha-1}}. \tag{4.3}$$

Since all players are symmetric, group $i$'s effort is $n_i$ times the effort of player $ik$. Multiplying both sides for $n_i$ we have

$$X_i = (\frac{X_j}{X^2})^{\frac{1}{\alpha-1}} n_i^{\frac{\alpha-2}{\alpha-1}}.$$

In order to calculate the group probability of winning and to simplify the following calculations, let us rewrite this function as

$$X_i = (\frac{X_j X_i}{X^2})^{\frac{1}{\alpha}} n_i^{\frac{\alpha-2}{\alpha}} \tag{4.4}$$

The uniqueness of the equilibrium follows from Theorem 3 of Cornes and Hartley (2005) through the theoretical quantity called share function, which allows us to make the analysis transparent. Define $\sigma_i = X_i/X$, and $(1-\sigma_i) = X_j/X$.

The group $i$ total effort can be defined as:

$$X_i = (\sigma_i(1-\sigma_i))^{1/\alpha} n_i^{\frac{\alpha-2}{\alpha}} \tag{4.5}$$

**Proposition 3 Cornes and Hartley (2005)**

If Assumption 1 holds for contestant $ik$ the share function $\sigma_i(X)$ has the following properties: $X^* > 0$ is the total amount of effort in a pure strategy Nash Equilibrium if and only if $\sum \sigma_i(X) = 1$. For each group the equilibrium effort level denoted with $X_i^* > 0$ is derived from the equation $\sigma_i(X) = \frac{X_i}{X}$. Furthermore,

1) $\sigma_i(X)$ is continuous with respect to $X > 0$;

2) $\sigma_i(X)$ is strictly decreasing as long as $X > 0$;

3) $lim_{X\to\infty}\sigma_i(X) = 0$ and $lim_{X\to 0}\sigma_i(X) = 1$.

By the Intermediate Value Theorem, these properties in conjunction with Assumption 1 establish the existence and uniqueness of a pure-strategy Nash equilibrium. Since $0 < \sigma_i(X^*) < 1$ for each $i$, all groups in equilibrium exert positive effort. Therefore, at least a player in each group exerts positive effort.

Typically, it is not possible to write down an explicit function for $\sigma_i$. However, in a competition between two groups, we are able to directly calculate $\sigma_i = \frac{X_i}{X}$ and get an explicit functional form.

In a two groups competition, the share functions (i.e., the probabilities of winning) for groups $a$ and $b$ are:

$$\sigma_a^* = \frac{n_a^{\frac{\alpha-2}{\alpha}}}{n_a^{\frac{\alpha-2}{\alpha}} + n_b^{\frac{\alpha-2}{\alpha}}} \text{ and } \sigma_b^* = \frac{n_b^{\frac{\alpha-2}{\alpha}}}{n_b^{\frac{\alpha-2}{\alpha}} + n_a^{\frac{\alpha-2}{\alpha}}}.$$

Clearly, $0 < \sigma_a < 1$, $0 < \sigma_b < 1$ and $\sigma_a + \sigma_b = 1$.

The total effort of group $i$ is:

$$X_i^* = (\sigma_i^*(1 - \sigma_i^*))^{1/\alpha} n_i^{\frac{\alpha-2}{\alpha}}$$

We notice that for $1 < \alpha < 2$ the small group overperforms the larger one. For $\alpha = 2$ the number of group members does not affect the probability of winning. For $\alpha > 2$ the larger group overperforms the small one. The latter case was firstly proved by Estaban and Ray (2001).

*Proof.* Consider groups $a$ and $b$ that compete in a contest. The $i$'s total effort is given by equation (4.5). Thus we have:

$$X_a = (\sigma_a(1-\sigma_a)^{1/\alpha} n_a^{\frac{\alpha-2}{\alpha}} \text{ and } X_b = (\sigma_b(1-\sigma_b)^{1/\alpha} n_b^{\frac{\alpha-2}{\alpha}}$$

The share function for group $a$ is $\sigma_a = X_a/(X_a + X_b)$. That is,

$$\sigma_a^* = \frac{(\sigma_a(1-\sigma_a))^{1/\alpha} n_a^{\frac{\alpha-2}{\alpha}}}{(\sigma_a(1-\sigma_a))^{1/\alpha} n_a^{\frac{\alpha-2}{\alpha}} + (\sigma_b(1-\sigma_b))^{1/\alpha} n_b^{\frac{\alpha-2}{\alpha}}}$$

Since $\sigma_a + \sigma_b = 1$, we can rewrite $1 - \sigma_a = \sigma_b$ and $1 - \sigma_b = \sigma_a$.

$$\sigma_a^* = \frac{(\sigma_a \sigma_b)^{1/\alpha} n_a^{\frac{\alpha-2}{\alpha}}}{(\sigma_a \sigma_b)^{1/\alpha} n_a^{\frac{\alpha-2}{\alpha}} + (\sigma_b \sigma_a)^{1/\alpha} n_b^{\frac{\alpha-2}{\alpha}}}$$

A simple reformulation yield our explicit share function:

$$\sigma_a^* = \frac{n_a^{\frac{\alpha-2}{\alpha}}}{n_a^{\frac{\alpha-2}{\alpha}} + n_b^{\frac{\alpha-2}{\alpha}}}$$

□

The best response given by Equation (4.3) can be thought as the supply of effort of each player at any given equilibrium point. This function has constant elasticity equal to $\epsilon = \frac{1}{\alpha-1}$. The elasticity of the best response function with respect to the prize is elastic for $1 < \alpha < 2$, unitarian for $\alpha = 2$, and anelastic for $\alpha > 2$. Considering that aggregate group effort is the linear sum of players' $x_{ik}$, efforts of group members are perfect substitutes. Hence, when the elasticity with respect to the prize is $0 < \epsilon < 1$ ($\alpha > 2$) an increase in the number of players increases group aggregate effort, and in turn the probability of winning. The converse occurs for $\epsilon > 1$.

In other words, every new player that joins a group leads to a decrease in the prize share of all other members. As a consequence, these players decrease their contribution of effort.

However, for $0 < \epsilon < 1$, the decrease of contributions is more than compensated by the effort of the newcomer. The sufficient and necessary condition is $\alpha > 2$. In the case of quadratic cost of effort, the best response has a unitarian elasticity. Since all players are identical, for every added player, the decrease of effort of all other members is equally compensated by the newcomers.

A first intuition can be gathered from this analysis. Assume that a manager can decide the size of his own group. If the costs of effort are not too steep, $1 < \alpha \leq 2$, the manager maximizes the probability of winning competing with a single player. Otherwise, the probability of winning increases with the increase of group size. However, this simple analysis focuses on the effect of group members under the assumption of identical players. In the next section we relax this restriction to analyze the effect of heterogeneity in abilities among players, while keeping the number of group members fixed.

## 4.3 Equal prize allocation among heterogeneous players

In this section we analyze the equal allocation of the prize in a noncooperative contest where players have heterogeneous ability parameters. This analysis is useful for two reasons. First, we aim to predict the equilibrium of the contest. Second, assuming that the group's structure is not exogeneously given, we attempt to provide a benchmark for sorting players that maximizes the overall effort exerted.

For instance, salespeople often work in teams and compete in tournaments. Clearly, both teams and tournaments are set up to increase employees' productivity. Thus, we provide a solution that allows the organizer to maximize aggregate effort via the sorting of players. Our result is in line with the intuition that in balanced competitions agents exert higher effort.

Differently from the preceding section, we now assume that $v_{ik}$ may vary among players and groups are composed of $n_k$ players that equally share the prize, $\varphi_i = \frac{1}{n_i}$.

Player $ik$'s expected payoff is

$$\pi_{ik} = \frac{X_i}{X}\frac{1}{n_i} - v_{ik}^{-1}x_{ik}^{\alpha}$$

The best response for player $ik$ is

$$x_{i1} = (\frac{X_j}{X^2}\frac{v_{ik}}{\alpha n_1})^{\frac{1}{\alpha-1}}, \tag{4.6}$$

Given the optimal responses for all players in group $i$, the group effort $X_i = \sum x_{ik}$ is defined as

$$X_i = (\frac{(1-\sigma_i)\sigma_i}{\alpha}\frac{1}{n_i})^{1/\alpha}(\sum v_{ik}^{\frac{1}{\alpha-1}})^{\frac{\alpha-1}{\alpha}} \tag{4.7}$$

*Proof.* The group total effort is simply the sum of individual efforts. For instance, the linear sum of optimal responses given by equation (4.6) in a group of two players is

$$X_i = (\frac{X_j}{X^2}\frac{v_{i1}}{2\alpha})^{\frac{1}{\alpha-1}} + (\frac{X_j}{X^2}\frac{v_{i2}}{2\alpha})^{\frac{1}{\alpha-1}}.$$

Rearranging, we have

$$X_i = (\frac{X_j}{X^2}\frac{1}{2\alpha})^{1/\alpha-1}(v_{i1}^{\frac{1}{\alpha-1}} + v_{i2}^{\frac{1}{\alpha-1}})$$

In order to simplify the analysis, we multiply both sides for $X_i^{\frac{1}{\alpha-1}}$ and define the group total effort as:

$$X_i = (\frac{X_jX_i}{X^2}\frac{1}{2\alpha})^{1/\alpha}(v_{i1}^{\frac{1}{\alpha-1}} + v_{i2}^{\frac{1}{\alpha-1}})^{\frac{\alpha-1}{\alpha}}$$

to finally get

$$X_i = ((1-\sigma_i)\sigma_i \frac{1}{2\alpha})^{1/\alpha}(v_{i1}^{\frac{1}{\alpha-1}} + v_{i2}^{\frac{1}{\alpha-1}})^{\frac{\alpha-1}{\alpha}}$$

The extension to $n_i$ players is straightforward.

□

The share function can be calculated via $\sigma_i = \frac{X_i}{X}$. In a contest between two groups, it can be determined as:

$$\sigma_i^* = \frac{(\sum v_{ik}^{\frac{1}{\alpha-1}})^{\frac{\alpha-1}{\alpha}}}{(\sum v_{ik}^{\frac{1}{\alpha-1}})^{\frac{\alpha-1}{\alpha}} + (\sum v_{jk}^{\frac{1}{\alpha-1}})^{\frac{\alpha-1}{\alpha}}}$$

**Proposition 4.3.1.** *In the noncooperative model, given two groups with the same mean in ability parameters that equally share the prize among members, we have that:*

- *for* $1 < \alpha < 2$ *the group with highest variance in ability among players has a higher winning probability;*
- *for* $\alpha = 2$ *groups have equal probability of winning;*
- *for* $\alpha > 2$ *the most homogeneous group in term of ability has a higher winning probability.*

*Proof.* Consider two groups $a$ and $b$ composed of two players. The share functions are defined as:

$$\sigma_a = \frac{(v_{a1}^{\frac{1}{\alpha-1}} + v_{a2}^{\frac{1}{\alpha-1}})^{\frac{\alpha-1}{\alpha}}}{(v_{a1}^{\frac{1}{\alpha-1}} + v_{a2}^{\frac{1}{\alpha-1}})^{\frac{\alpha-1}{\alpha}} + (v_{b1}^{\frac{1}{\alpha-1}} + v_{b2}^{\frac{1}{\alpha-1}})^{\frac{\alpha-1}{\alpha}}}$$

$$\sigma_b = \frac{(v_{b1}^{\frac{1}{\alpha-1}} + v_{b2}^{\frac{1}{\alpha-1}})^{\frac{\alpha-1}{\alpha}}}{(v_{b1}^{\frac{1}{\alpha-1}} + v_{b2}^{\frac{1}{\alpha-1}})^{\frac{\alpha-1}{\alpha}} + (v_{a1}^{\frac{1}{\alpha-1}} + v_{a2}^{\frac{1}{\alpha-1}})^{\frac{\alpha-1}{\alpha}}}$$

Notice that $0 < \sigma_i < 1$, for $i = a, b$.

Without loss of generality, assume $v_{a1} + v_{a2} = v_{b1} + v_{b2}$, $v_{a1} = v_{a2}$ and $v_{b1} \neq v_{b2}$. We have that $\sigma_a \lesseqgtr \sigma_b$ gives

$$(v_{a1}^{\frac{1}{\alpha-1}} + v_{a2}^{\frac{1}{\alpha-1}})^{\frac{\alpha-1}{\alpha}} \lesseqgtr (v_{b1}^{\frac{1}{\alpha-1}} + v_{b2}^{\frac{1}{\alpha-1}})^{\frac{\alpha-1}{\alpha}}$$

Rearranging we get:

$$2v_{a1}^{\frac{1}{\alpha-1}} \lesseqgtr v_{b1}^{\frac{1}{\alpha-1}} + v_{b2}{}^{\frac{1}{\alpha-1}} \tag{4.8}$$

□

As for the preceding section, if the cost function is sufficiently steep, $\alpha > 2$, the best response function is concave. This implies that, keeping the number of players fixed, for the same mean in ability parameters across groups, a homogeneous distribution leads to a higher aggregate effort. Conversely, for a sufficiently low steepness of the cost function, $1 < \alpha < 2$, the best response function is convex. Hence, a heterogeneous distribution of abilities increases the aggregate effort.

In the next section we embody both the effect of group size and heterogeneity among players to study the optimal sorting.

### 4.3.1 Optimal sorting of players among groups

In this section we study the optimal sorting of players that maximizes aggregate effort. We assume that players can be sorted into two groups. Moreover, members equally share the prize. In this environment groups can differ in size.

Equation (4.7) defines total group $i$'s effort in equilibrium. That is, in a two group contest the total effort of group $a$ is

$$X_a^* = (\frac{\sigma_b \sigma_a}{\alpha} \frac{1}{n_a})^{1/\alpha} (\sum v_{ak}^{\frac{1}{\alpha-1}})^{\frac{\alpha-1}{\alpha}} \tag{4.9}$$

and respectively for group $b$ is

$$X_b^* = (\frac{\sigma_a \sigma_b}{\alpha} \frac{1}{n_b})^{1/\alpha} (\sum v_{bk}^{\frac{1}{\alpha-1}})^{\frac{\alpha-1}{\alpha}}. \tag{4.10}$$

The overall effort $X$ is the sum of the group's efforts. A simple reformulation yields

$$X^* = (\frac{\sigma_a \sigma_b}{\alpha})^{1/\alpha} (\frac{\sum v_{ak}^{\frac{1}{\alpha-1}}}{n_a^{\frac{1}{\alpha-1}}} + \frac{\sum v_{bk}^{\frac{1}{\alpha-1}}}{n_b^{\frac{1}{\alpha-1}}})^{\frac{\alpha-1}{\alpha}}.$$

All parameters $v_{ik}$ are exogenously given, and aggregate effort depends only on $\sigma_a \sigma_b$ that can be manipulated through the sorting of players. It follows that $X^*$ is maximized for $\sigma_a = \sigma_b = 1/2$, that is when a competition is perfectly balanced.

**Proposition 4.3.2.** *The optimal sorting of players equalizes probability of winning. It is given by*

$$\frac{\sum v_{ak}^{\frac{1}{\alpha-1}}}{n_a^{\frac{1}{\alpha-1}}} = \frac{\sum v_{bk}^{\frac{1}{\alpha-1}}}{n_b^{\frac{1}{\alpha-1}}}$$

The optimal sorting of players depends on ability parameters, cost steepness (see Proposition 4.3.1) and group size. Given the same players, but different cost convexity, we may have different optimal sorting. This result allows to sort players in order to make competitions more balanced, that in turn leads to a higher aggregate effort.
(Another important implication of proposition (4.3.2) is that, by allowing heterogeneity, the statement that for $\alpha > 2$ larger groups overperform small ones no longer holds.)

## 4.4 Optimal prize allocations

In previous sections we have seen that larger groups overperform smaller ones for a sufficiently high steepness of the cost function, $\alpha > 2$. We have relaxed the assumption of identical players to analyze the effects of heterogeneity on group effort. Even in that case, our result depends on the degree of cost convexity. Common to both analysis is the threshold $\alpha = 2$ over which results are reversed. As we are going to see, for the optimal prize allocation we have the same threshold.

In this section we provide a benchmark to optimally allocate the prize in groups composed of two players.

Assume $1 < \alpha \leq 2$. Since the prize allocation is not exogenously given, $\varphi_i$ is our variable of interest. The optimal responses for any value of $\varphi_i$ are given by the maximization of the expected utilities of groups members.

Player $ik$'s expected payoff is:

$$\pi_{ik} = \frac{X_i}{X}\varphi_i - v_{ik}x_{ik}^{\alpha}$$

Player $i1$'s best response is:

$$x_{i1} = (\frac{X_{j\neq i}}{X^2}\varphi_i v_{i1})^{\frac{1}{\alpha-1}} \tag{4.11}$$

Player $i2$'s best response is

$$x_{i2} = (\frac{X_{j\neq i}}{X^2}(1-\varphi_i)v_{i1})^{\frac{1}{\alpha-1}} \tag{4.12}$$

The aggregate effort of group $i$ is $X_i = x_{i1} + x_{i2}$ and corresponds to:

$$X_i = (\frac{(1-\sigma_i)\sigma_i}{\alpha})^{1/\alpha}((\varphi_i v_{i1})^{\frac{1}{\alpha-1}} + ((1-\varphi_i)v_{i2})^{\frac{1}{\alpha-1}})^{\frac{\alpha-1}{\alpha}}$$

Therefore, $\sigma_i$ can be uniquely determined as $\sigma_i = \frac{X_i}{X}$, yielding

$$\sigma_i^* = \frac{[(\varphi_i v_{i1})^{\frac{1}{\alpha-1}} + ((1-\varphi_i)v_{i2})^{\frac{1}{\alpha-1}}]^{\frac{\alpha-1}{\alpha}}}{[(\varphi_i v_{i1})^{\frac{1}{\alpha-1}} + ((1-\varphi_i)v_{i2}^{\frac{1}{\alpha-1}}]^{\frac{\alpha-1}{\alpha}} + [(\varphi_j v_{j1})^{\frac{1}{\alpha-1}} + ((1-\varphi_j)v_{j2})^{\frac{1}{\alpha-1}}]^{\frac{\alpha-1}{\alpha}}} \tag{4.13}$$

We can now state how a team manager should allocate the prize in order to maximize the group's probability of winning for $1 < \alpha \leq 2$:

**Proposition 4.4.1.**

- *If* $v_{i1} > v_{i2}$ *then* $\varphi_i = 1$.
- *If* $v_{i1} < v_{i2}$ *then* $\varphi_i = 0$.
- *If* $v_{i1} = v_{i2}$ *then* $\varphi = 0$ *or* $\varphi_i = 1$ *are optimal solutions.*

A particular case occurs for $\alpha = 2$ and $v_{i1} = v_{i2}$, when any value $\varphi_i = [0,1]$ of the prize allocation is indifferent. Except for the latter case, the probability of winning of team $i$ can be always enhanced allocating the prize differently from the equal division. Moreover, if both team managers apply $\varphi_i^*$, the contest among two groups is equivalent to a contest among two individuals.

*Proof.* Consider a contest between two groups labelled with $a$ and $b$. The function (4.13) can be rewritten as follows:

$$\sigma_a = \frac{f(\varphi_a)}{f(\varphi_a) + f(\varphi_b)}$$

In order to maximize this function is sufficient to maximize $f(\varphi_a)$. To see why, just rewrite $\sigma_a$ as

$$\sigma_a = \frac{f(\varphi_a) + f(\varphi_b) - f(\varphi_b)}{f(\varphi_a + f(\varphi_b)} = \sigma_a = 1 - \frac{f(\varphi_b)}{f(\varphi_a) + f(\varphi_b)}$$

Since $f(\varphi_b) > 0$, $\sigma_a$ is increasing in $f(\varphi_a)$. Moreover, for $1 < \alpha < 2$ the function $f(\varphi_a)$ is strictly convex. It follows that the solution that maximizes $\sigma_a$ is $\varphi_a = 1$ or $\varphi_a = 0$. □

In section (4.2) we have investigated the effect on group effectiveness of an increase or decrease in group size. We have shown that if the prize elasticity of the best response function is $\epsilon > 1$ $(1 < \alpha < 2)$, an increase in group size decreases aggregate effort. In this environment, we keep the number of group members fixed, thus the manager can only allocate the prize to increase group effectiveness. The result is in line with the preceding one. Indeed, even in that case, for a response of effort elastic with respect to the prize it is optimal to allocate the entire prize to a single player. Moreover, notice that this is equivalent to reducing the number of active players in the contest for a given group. If a player does not receive a positive part of the prize, clearly, he does not exert effort. Hence, under the assumption of identical players, even if $1 < \alpha < 2$, larger groups have at least an equal probability of winning than smaller ones. However, allocating the whole prize to a single player or competing with a singleton are not equivalent solution. Consider a prize composed both by private and public characteristics. The private prize should be allocated to a single player. However, due to the public part, all other members are active in equilibrium and exert positive effort. Thus, a large group overperform the smaller one under the symmetry assumption, for any cost steepness. If the prize is a pure private good, groups have equal probability of winning.

Assume that $\alpha > 2$. In that case, an increase in the prize share for a player increases his effort at a decreasing rate. It follows that a unique optimal prize allocation exists. Moreover all players are active in the contest even if the prize is a pure private good.
For any $\alpha > 2$ the optimal prize allocation is

$$\varphi_i^* = \frac{v_{i1}^{\frac{1}{\alpha-2}}}{v_{i1}^{\frac{1}{\alpha-2}} + v_{i2}^{\frac{1}{\alpha-2}}}$$

Notice that for $\alpha > 2$ both players receive a positive share of the prize. Moreover, for $\alpha \to \infty$ we have $\varphi_i^* \to 1/2$.

*Proof.* Recall that the probability of winning can be rewritten as

$$\sigma_a = 1 - \frac{f(\varphi_a)}{f(\varphi_a) + f(\varphi_b)}$$

Since $f(\varphi_b)$ is always positive, $\sigma_a$ is increasing in $f(\varphi_a)$. Moreover, for $\alpha > 2$ the function $f(\varphi_a)$ is strictly concave. It follows that the solution is unique and is given by the first derivative of $f(\varphi_a)$ with respect to $\varphi_a$. That is,

$$f'(\varphi_a)\left(\frac{1}{\alpha-1}\frac{(v_{a1}\varphi_a)^{\frac{1}{\alpha-1}}}{\varphi_a} - \frac{1}{\alpha-1}\frac{[v_{a2}(1-\varphi_i)]^{\frac{1}{\alpha-1}}}{(1-\varphi_i)}\right) = 0$$

Rearranging, we get

$$\frac{(v_{a1}\varphi_a)^{\frac{1}{\alpha-1}}}{\varphi_a} = \frac{(v_{a2}(1-\varphi_a))^{\frac{1}{\alpha-1}}}{1-\varphi_a}$$

and finally

$$\varphi_a^* = \frac{v_{a1}^{\frac{1}{\alpha-2}}}{v_{a1}^{\frac{1}{\alpha-2}} + v_{a2}^{\frac{1}{\alpha-2}}}$$

□

If the prize elasticity of best responses is $\epsilon < 1$ (i.e. the best responses functions are concave), for each increasing rate of prize assigned to a players, the quantity of effort exerted increases at a decreasing level. Hence, it is more efficient to allocate both players with a positive share of the prize.

Let us interpret this result. Consider the effort $x_{ik}$ as the time spent by an employee working on a project with an uncertain outcome. The higher the time devoted to the project, the more is the marginal value of the remaining time. It follows that, for an increase in prize share, an additional hour of work must be compensated by rewarding the players with a larger part of the prize with respect to the preceding hour. However, for the same increase in prize, the other players dedicate more time to work, given that his value for the remaining hours is lower then the other player. This reasoning should continue until the whole prize is allocated.

In order to conclude the analysis of noncooperative contests, we state which group has a higher probability of winning if both of them apply $\varphi^*$.

**Proposition 4.4.2.** *Given that both teams apply $\varphi_i^*$ we have that:*

- *for $2 < \alpha < 3$ the group with the highest variance in ability has a higher winning probability;*
- *for $\alpha = 3$ groups have equal probability of winning;*
- *for $\alpha > 3$ the most homogeneous group in term of ability has higher winning probability.*

*Proof.* Consider a contest between group $a$ and group $b$. The share functions are defined as follows:

$$\sigma_a^* = \frac{[(\varphi_a v_{a1})^{\frac{1}{\alpha-1}} + ((1-\varphi_a)v_{a2})^{\frac{1}{\alpha-1}}]^{\frac{\alpha-1}{\alpha}}}{[(\varphi_a v_{i1})^{\frac{1}{\alpha-1}} + ((1-\varphi_a)v_{a2}^{\frac{1}{\alpha-1}}]^{\frac{\alpha-1}{\alpha}} + [(\varphi_b v_{b1})^{\frac{1}{\alpha-1}} + ((1-\varphi_b)v_{b2})^{\frac{1}{\alpha-1}}]^{\frac{\alpha-1}{\alpha}}}$$

$$\sigma_b^* = \frac{[(\varphi_b v_{b1})^{\frac{1}{\alpha-1}} + ((1-\varphi_b)v_{b2})^{\frac{1}{\alpha-1}}]^{\frac{\alpha-1}{\alpha}}}{[(\varphi_b v_{b1})^{\frac{1}{\alpha-1}} + ((1-\varphi_b)v_{b2}^{\frac{1}{\alpha-1}}]^{\frac{\alpha-1}{\alpha}} + [(\varphi_a v_{a1})^{\frac{1}{\alpha-1}} + ((1-\varphi_a)v_{a2})^{\frac{1}{\alpha-1}}]^{\frac{\alpha-1}{\alpha}}}$$

We can check that $\sigma_a \lesseqgtr \sigma_b$ depending on

$$[(\varphi_a v_{a1})^{\frac{1}{\alpha-1}} + ((1-\varphi_a)v_{a2})^{\frac{1}{\alpha-1}}]^{\frac{\alpha-1}{\alpha}} \lesseqgtr [(\varphi_b v_{b1})^{\frac{1}{\alpha-1}} + ((1-\varphi_b)v_{b2})^{\frac{1}{\alpha-1}}]^{\frac{\alpha-1}{\alpha}}$$

Replacing $\varphi_a$ with $\varphi_a^*$ (if and only if $\alpha > 2$) after some arrangements we get

$$v_{i1}^{\frac{1}{\alpha-2}} + v_{i2}^{\frac{1}{\alpha-2}} \lesseqgtr v_{j1}^{\frac{1}{\alpha-2}} + v_{j1}^{\frac{1}{\alpha-2}}$$

Without loss of generality assume that $v_{i1} + v_{i2} = v_{j1} + v_{j2}$, $v_{i1} = v_{i2}$ and $v_{j1} \neq v_{j2}$. We finally have

$$2v_{a1}^{\frac{1}{\alpha-2}} \lesseqgtr v_{b1}^{\frac{1}{\alpha-2}} + v_{b2}{}^{\frac{1}{\alpha-2}}$$

□

## 4.5 Cooperative Model

In a cooperative model agents decide how much effort to expend simultaneously and co-operatively. In order to investigate cooperative contests, we assume that players consider cooperation the most fruitful individual strategy. We are not concerned with why this occurs. Indeed cooperation may be a consequence of organizational schemes direct to monitor agents behavior, as in sport competition, it may be the winning strategy. In all these cases, free-riding is less acute. However, even if we are sure that players cooperate, our assumption remains “ad hoc”. There is no reason why agents fully cooperate with other group members. While perfect cooperation is always uncertain, to investigate such case we assume full cooperation within groups.

If the group members cooperate, the expected payoff for player $ik$ is:

$$\pi_{ik} = \frac{X_i}{X}\varphi_{ik} - v_{ik}^{-1} g(X_i s_{ik})$$

where $X_i s_{ik} = x_{ik}$ and $s_{ik}$ corresponds to the share of the group effort that player $ik$ exerts in equilibrium. Hence, the player's effort $x_{ik}$ is rewritten as a part of the total effort exerted by the group. For instance, if players in the groups are identical, $s_{ik} = 1/n$.

In the cooperative contest player's choices of effort are given by the maximization of $\pi_{ik}$ as a function of $X_i$. In a group of two players, player $i1$'s best response is:

$$g'(X_i s_i) = \frac{X_j}{X^2}\frac{v_{i1}\varphi_i}{\alpha} \tag{4.14}$$

and player $i2$'s best response is:

$$g'(X_i(1 - s_i)) = \frac{X_j}{X^2}\frac{v_{i2}(1 - \varphi_i)}{\alpha} \tag{4.15}$$

Under Assumption 1, Equations (4.14) and (4.15) are both necessary and sufficient for maximization.

We are interested in studying the effect of prize allocations on group effectiveness. Clearly, allocating the whole prize to a single player leads the other group member to exert zero effort. In noncooperative contests, we have shown that, under some conditions, this unequal allocation is efficient in term of group effectiveness. Here we are in a different environment, where the preceding propositions may not be still valid. Indeed, we show how the cooperation among players strongly influences both the manager prize allocation decision and the groups effectiveness, leading to different results.

We begin our analysis investigating the effect of the number of group members on the probability of winning.

## 4.6 Group size effect in cooperative contests

Consider two competing groups composed of $n_i$ players. All players are symmetric in their ability parameter, $v_{ik} = 1$, and are rewarded with an equal share of the prize, $\varphi_i = 1/n_i$. Moreover, the cost of expending effort is $g(X_i s_{ik}) = (X_i s_{ik})^\alpha$. The best response for player $ik$ is:

$$X_i^{\alpha-1} s_{ik}^\alpha = \frac{X_j}{X^2}\frac{n_i}{\alpha}.$$

Multiplying both sides for $X_i$ and rearranging, we get player $ik$'s choice of effort

$$X_i^\alpha s_{ik}^\alpha = \frac{X_j X_i}{X^2}\frac{n_i}{\alpha}.$$

As for noncooperative contests, the unique equilibrium of the game and the aggregate group effort are easily identified via the share function, $\sigma_i = \frac{X_i}{X}$. That is,

$$x_{ik} = X_i s_{ik} = (\frac{(1-\sigma_i)\sigma_i}{\alpha n_i})^{\frac{1}{\alpha}} \tag{4.16}$$

Since players are symmetric, multiplying both sides for $n_i$, we get the total effort of group $i$

$$X_i = (\frac{(1-\sigma_i)\sigma_i}{\alpha n_i})^{\frac{1}{\alpha}} n_i. \tag{4.17}$$

In a two groups contest, the share function for group $i$ is:

$$\sigma_i^* = \frac{n_i^{\frac{\alpha-1}{\alpha}}}{n_i^{\frac{\alpha-1}{\alpha}} + n_j^{\frac{\alpha-1}{\alpha}}} \tag{4.18}$$

*Proof.* We can rewrite equation (4.17) as:

$$X_i = (\frac{(1-\sigma_i)\sigma_i}{\alpha})^{\frac{1}{\alpha}} n_i^{\frac{\alpha-1}{\alpha}}$$

In a two groups competition, the share function for $a$ is:

$$\sigma_a = \frac{(\frac{(1-\sigma_a)\sigma_a}{\alpha})^{\frac{1}{\alpha}} n_i^{\frac{\alpha-1}{\alpha}}}{(\frac{(1-\sigma_a)\sigma_a}{\alpha})^{\frac{1}{\alpha}} n_i^{\frac{\alpha-1}{\alpha}} + (\frac{(1-\sigma_b)\sigma_b}{\alpha})^{\frac{1}{\alpha}} n_i^{\frac{\alpha-1}{\alpha}}}$$

Since $\sigma_a + \sigma_b = 1$, we rewrite $1 - \sigma_a = \sigma_b$ and $1 - \sigma_b = \sigma_a$. A simple reformulation yields:

$$\sigma_a = \frac{n_a^{\frac{\alpha-1}{\alpha}}}{n_a^{\frac{\alpha-1}{\alpha}} + n_b^{\frac{\alpha-1}{\alpha}}}$$

□

In equilibrium the total effort of group $i$ is:

$$X_i^* = (\frac{(1-\sigma_i^*)\sigma_i^*}{\alpha})^{\frac{1}{\alpha}} n_i^{\frac{\alpha-1}{\alpha}}$$

We notice from Equation (4.18) that for any $\alpha > 1$ larger groups overperform small ones. Hence, smaller groups never outperform larger ones in a cooperative environment since each additional player increases group performance. Indeed, notice that from Equation (4.16) the elasticity of response functions with respect to the prize is $0 < \epsilon < 1$ for any value of $\alpha$. This implies that the best response function is always concave and that a decrease in prize is always more than compensated by the effort exerted by the newcomer. However, such result clearly depends on the perfect cooperation assumption. In the next section we investigate the effect of heterogeneity in ability among players, while keeping fixed the number of members in groups.

## 4.7 Equal prize allocation among cooperative players

In this section we analyze the equilibrium of a cooperative game where groups are composed of two players that equally share the prize and have heterogeneous ability parameters. Thus $v_{ik}$ may vary among agents and the prize $\varphi_i = 1/2$.

Player $ik$'s expected payoff is

$$\pi_{ik} = \frac{X_i}{X}\frac{1}{2} - v_{ik}^{-1}(X_i s_{ik})^{\alpha}$$

The best response for player $i1$ and $i2$ are respectively,

$$x_{i1} = X_i s_i = (\tfrac{(1-\sigma_i)\sigma_i}{\alpha}\tfrac{v_{i1}}{2})^{\frac{1}{\alpha}} \text{ and } x_{i2} = X_i(1-s_i) = (\tfrac{(1-\sigma_i)\sigma_i}{\alpha}\tfrac{v_{i2}}{2})^{\frac{1}{\alpha}}.$$

The aggregate effort of group $i$ is the linear sum of the two optimal responses, and it corresponds to

$$X_i = (\frac{(1-\sigma_i)\sigma_i}{\alpha 2})^{\frac{1}{\alpha}}(v_{i1}^{\frac{1}{\alpha}} + v_{i2}^{\frac{1}{\alpha}})$$

In a contest between two groups, the share function is determined as:

$$\sigma_i^* = \frac{(v_{i1}^{\frac{1}{\alpha}} + v_{i2}^{\frac{1}{\alpha}})}{(v_{i1}^{\frac{1}{\alpha}} + v_{i2}^{\frac{1}{\alpha}}) + (v_{j1}^{\frac{1}{\alpha}} + v_{j2}^{\frac{1}{\alpha}})} \tag{4.19}$$

**Proposition 4.7.1.** *In a cooperative contest, given two groups with the same mean in ability parameters that equally share the prize among members, the most homogeneous group has a higher winning probability.*

*Proof.* Consider groups $a$ and $b$ and their respective share functions given by equation (4.19). Without loss of generality, assume $v_{a1} + v_{a2} = v_{b1} + v_{b2}$, $v_{a1} = v_{a2}$ and $v_{b1} \neq v_{b2}$. We have that $\sigma_a \geq \sigma_b$ gives

$$(v_{a1}^{\frac{1}{\alpha}} + v_{a2}^{\frac{1}{\alpha}}) \geq (v_{b1}^{\frac{1}{\alpha}} + v_{b2}^{\frac{1}{\alpha}}).$$

Rearranging,

$$2v_{a1}^{\frac{1}{\alpha}} \geq (v_{b1}^{\frac{1}{\alpha}} + v_{b2}^{\frac{1}{\alpha}})$$

□

In cooperative environments the free-riding problem does not occur. The optimal response functions of players are concave for any $\alpha > 1$. Thus, given the same mean in ability parameter homogeneous groups outperform heterogeneous ones.

## 4.8 Optimal prize allocation in cooperative contests

In this framework, since the prize allocation is not exogenously given, $\varphi_i$ is our variable of interest. Rearranging players' best responses we get:

$$x_{i1} = X_i s_i = (\frac{(1-\sigma_i)\sigma_i}{\alpha} v_{i1}\varphi_i)^{\frac{1}{\alpha}}$$

$$x_{i2} = X_i(1-s_i) = (\frac{(1-\sigma_i)\sigma_i}{\alpha} v_{i2}(1-\varphi_i))^{\frac{1}{\alpha}}$$

The aggregate effort $X_i$ is determined by the linear sum of efforts, and corresponds to:

$$X_i = (\frac{(1-\sigma_i)\sigma_i}{\alpha})^{\frac{1}{\alpha}}[(v_{i1}\varphi_i)^{\frac{1}{\alpha}} + (v_{i2}(1-\varphi_i))^{\frac{1}{\alpha}}]$$

Therefore, $\sigma_i$ can be determined as $\sigma_i = \frac{X_i}{X}$, yielding

$$\sigma_i^* = \frac{[(v_{i1}\varphi_i)^{\frac{1}{\alpha}} + (v_{i2}(1-\varphi_i))^{\frac{1}{\alpha}}]}{[(v_{i1}\varphi_i)^{\frac{1}{\alpha}} + (v_{i2}(1-\varphi_i))^{\frac{1}{\alpha}}] + [(v_{j1}\varphi_j)^{\frac{1}{\alpha}} + (v_{j2}(1-\varphi_j))^{\frac{1}{\alpha}}]} \tag{4.20}$$

The optimal prize allocation is:

$$\varphi_i^* = \frac{v_{i1}^{\frac{1}{\alpha-1}}}{v_{i1}^{\frac{1}{\alpha-1}} + v_{i2}^{\frac{1}{\alpha-1}}}$$

**Proposition 4.8.1.** *In the cooperative contests both players should be rewarded with a strictly positive share of the prize for any $\alpha > 1$. Moreover, as $\alpha \to \infty$ we have $\varphi^* \to 1/2$.*

*Proof.* As in the proof of noncooperative contests, in order to maximize group probability of winning it is sufficient to maximize the numerator of Equation (4.20). The first derivative with respect to $\varphi_i$ is

$$\frac{(v_{i1}\varphi_i)^{\frac{1}{\alpha}}}{\varphi_i} = \frac{(v_{i2}(1-\varphi_i))^{\frac{1}{\alpha}}}{1-\varphi_i}.$$

Rearranging, we get

$$v_{i1}^{\frac{1}{\alpha-1}}(1-\varphi_i) = v_{i2}^{\frac{1}{\alpha-1}}\varphi_i$$

and finally

$$\varphi_i^* = \frac{v_{i1}^{\frac{1}{\alpha-1}}}{v_{i1}^{\frac{1}{\alpha-1}} + v_{i2}^{\frac{1}{\alpha-1}}}$$

□

In order to conclude the analysis of cooperative contests, we state which group has higher winning probability if both of them apply $\varphi^*$.

**Proposition 4.8.2.** *Given that both teams apply $\varphi^*$, we have that:*

- *for $1 < \alpha < 2$ the group with the highest variance in ability has a higher probability of winning;*
- *for $\alpha = 2$ groups have equal probability of winning;*
- *for $\alpha > 2$ homogeneous group in term of ability overperform heterogeneous one.*

## 4.9 Do smaller groups outperform bigger ones?

Esteban and Ray (2001), under the restriction of identical players, show that the group size paradox is reversed for a sufficiently high cost steepness, $\alpha > 2$, that implies an inelastic response of effort with respect to the prize. Under this condition each newcomer increases group effectiveness, even if all other members, due to a decrease in individual rewards, reduce their effort. Conversely, for a low cost steepness, $1 < \alpha < 2$, the result is overturned. Our benchmark provides an optimal prize allocation that reverses the group size paradox even if best responses are elastic with respect to the prize, $\epsilon > 1$. Indeed, in this case, allocating the whole prize to a single player is more efficient because it decreases the number of active players in the group. That is, if all players are symmetric but groups differ in their size, under the optimal allocative rule the competition is equal to a contest between two players. As a consequence, recalling that players are identical, larger and smaller groups have at least equal probability of winning. We want to underline "at least". Indeed, if the prize is not a pure private good, for any strictly positive part of publicness, larger groups outperform smaller ones, even if $\epsilon > 1$. This conjecture is straightforward. Take a contest between a singleton and a group of two players that compete for a pure private prize. Applying the optimal allocative rule, we have that a player of the group does not exert effort. The competition is reduced to a contest between two identical players, and then the group and the singleton have equal probability of winning. Assume that the prize has a strictly positive part of publicness: keeping the same prize allocation, then the other player in the group exerts positive effort. Thus the group has a higher probability of winnng.

In sections (4.3) we investigated the effect of heterogeneity and homogeneity among group members. In our model, we considered the ability of players as a constant multiplicative parameter. We relaxed the assumption of identical players, and in section (4.3.1), we demonstrated that the effect of heterogeneity in ability may overturn the statement that, for a sufficiently higher cost steepness ($\alpha > 2$), larger groups overperform smaller ones. If larger groups have a higher variance in ability, even if the prize elasticity of best responses is inelastic, a group homogenous in term of ability may overperform the larger one.

We concluded our analysis of contest among groups, investigating the effect of cooperation among players. We did not compare cooperative groups with noncooperatives ones. Nevertheless, recent works[2] demonstrate that cooperation among players increase group effectiveness and leads cooperative groups to perform better than a noncooperative ones.

[2]See Ursprung (2012) and Cheikbossian (2006).

## 4.10 Conclusions

Our simple model focused on contests among two groups that compete for a prize in a one shot game. We are interested in analyzing the effect of heterogeneity among players on prize distribution. Moreover, we investigated which rules influence the optimal prize allocation. Our results depend on the steepness of the cost function, and in turn on the form of the best response functions. In our specific noncooperative framework, we found that: if the prize elasticity of best responses is elastic, the whole prize should be allocated to a single player (the most capable if they are heterogenous). Conversely, if the best response functions are inelastic with respect to the prize, both players should be rewarded with a positive share of the prize. The share assigned to players depends on their ability parameter. Moreover, the differences in prize allocation decrease as the cost convexity increases.

In our framework we introduced several strong assumptions: the competition takes place in a perfect information environment; the cost function has the form $x^{\alpha}$, which implies constant elasticity of best response function. Clearly, this restriction simplifies our analysis. Lastly, in order to optimally allocate the prize we assumed that groups are composed of two players.

A brief investigation was made regarding cooperative contests. The main difference with respect to the noncooperation case is that, due to the absence of free-riding, if costs are strictly convex, both players should be rewarded with a positive share of the prize.

Independently of the type of contest, we have shown the significance of heterogeneity among players. Moreover, we have shown the importance of prize allocation in group analysis. Indeed, if the prize is not optimally allocated, the analysis of group effectiveness is incomplete.

Our main goal was to find an allocative rule that maximizes group effectivenes, without paying attention to player expected payoff. We leave to future reasearch the study of an allocation of prize that is both efficient and fair.

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