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**Portfolio Value-at-Risk
and Expected-Shortfall
with Efficient
Simulation
Approaches**

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ABSTRACT

In many fields, particularly in the financial sector, risk management and control constitute an essential component for effective financial management and regulation. Management of risks often necessitates a holistic understanding of the risks borne by the financial institution and the measurement and communication of the risks to senior management. Thus, developing a powerful tool to efficiently and accurately measure risk is fundamental to making key financial and regulatory decisions; the more refined the risk estimation, the more accurate the predictive results and the more effective decisions based on the risk estimates. Risk speculation measures such as Value-at-Risk (VaR) and Expected-Shortfall (ES) have been predominantly used in financial risk assessment. However, evidence suggests that several of the approaches used in their computation produce estimates that are statistically far off their true values, and the simulation methods used are either slow or have more computational requirements, leading to longer computing efforts and inaccurate results. This thesis reviews VaR and ES, their calculation methods, and considers existing efficient simulation methods for their estimation to achieve faster computing speeds and more accurate results. It conducts an empirical study contrasting an Importance Sampling (IS) method based on a Gaussian mixture Model (GMM) of return distribution with an Expectation–Maximization (EM) algorithm for calibrating the model parameters with Geometric Brownian Motion (GBM) Monte Carlo methods.

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1 Introduction

In the financial sector, risk management or control constitutes an essential component for effective financial management and financial regulation. Management of risks often necessitates a holistic understanding of the risks borne by the financial institution as well as the measurement and communication of the borne risks to senior management. Thus, developing a powerful tool to efficiently and accurately measure risk is fundamental to making key financial and regulatory decisions; the more refined the estimation of risk, the more accurate the predictive results and the more effective decisions based on the risk estimates. Risk speculation measures such as Value-at-Risk (VaR) and Expected-Shortfall (ES) have been predominantly used in financial risk assessment. However, evidence suggests that several of the approaches used in their computation produce estimates that are statistically far off their true values, and the simulation methods used either slow or have more computational requirements, leading to longer computing efforts and inaccurate results. This project seeks to review VaR and ES calculation methods and to consider efficient simulation methods in the computation of these risk assessment measures that can lead to faster computing speeds and more accurate results.

VaR is one of the most popular measures for estimating market risk. However, it does not satisfy *coherence*, a basic property of the general class of risk measures. For that matter, researchers develop ES as a natural *coherent* substitute for VaR. Notwithstanding, both VaR and ES are still widely applied across several disciplines. Although many computation methods have been proposed for VaR and ES, most are time-consuming and are sometimes inaccurate. This project explores some simulation methods that ensure faster computation and more accurate estimates for ES and VaR.

VaR and ES are mainly concerned with market risk. Hence, this writing provides an overview of risk and market risk. It proceeds to give an in-depth description of VaR and ES, outlining their properties, presenting the methods for their computation, and stating their advantages and disadvantages. This thesis aims to employ simulation methods for efficiently computing VaR and ES. Thus, a brief introduction to simulation is given and the application areas of simulation are outlined.

The structure of the thesis is as follows. The second section describes risk and the motivation for applying VaR. The literature on VaR is described followed by an overview of the literature on ES. Section 5 introduces simulations and the sixth section delves into efficient simulation approaches. Section 7 briefly describes the methodology for estimating VaR and ES employing Geometric Brownian Motion (GBM) and details the Gaussian Mixture Modes (GMM) method for evaluating the risk measures. The final section provides the results from an empirical study that employs GBM Monte Carlo methods and GMM to calculate the VaR and ES of a portfolio of 15 FTSE MIB stocks.

2 Risk and Motivation for VaR

2.0.1 Risk

Industries and firms face the unavoidable burden of dealing with risks and their consequences. Although risks can pose horrendous effects on firms' outlook, they also provide an avenue for profit creation. [Jorion, 2007] also agrees with this opinion and asserts, "Whereas some firms accept risks passively, others attempt to create a competitive advantage by judicious exposure to risks. In both cases, however, the risks should be monitored carefully because of their potential for damage.". He defines risk as "the volatility of unexpected outcomes, which can represent the value of assets, equity, or earnings", and he classifies risks as belonging to two main categories, business risks, and financial risks. Business risks refer to the risks that firms and institutions intentionally take to gain competitive advantage while increasing the profitability of shareholders. Business risks involve strategic maneuvers concerning the decisions of management or executive board regarding marketing strategies, product development types, and the organizational structure of the business. It also involves the attitude of the firm towards macroeconomic risks or its tendency to take on macroeconomic risks. The other category of risk, which is of major concern in this project is financial risk.

Financial risk refers to the risks that accrue from financial market activities with the tendency to cause losses to firms or other business entities. For instance, exchange rate, interest rate volatilities, and default on financial obligations are risks that may cause losses to firms and hence pose tremendous challenges to firms. Financial risks may be disintegrated into four main types of risks, Credit risk, liquidity risk, operational risk, and market risk. Credit risk refers to the risk of potential default on financial obligations by

counterparties. It also includes sovereign risk which generally occurs when a country's monetary policy imposes foreign exchange controls making it difficult for counterparties to fulfill their obligations. Liquidity risk is of two forms, asset/market-liquidity risk and cash-flow/funding-liquidity risk. Asset-liquidity risk denotes the inability of a transaction to take place at prevailing market prices due to the size of the transaction relative to normal trading volumes, while funding-liquidity risk refers to the inability of entities to timely fulfill payment obligations leading to forced liquidation and the transformation of paper losses to realized losses. Operational risk refers to the risk of losses accruing from ineffective or flawed internal procedures, individuals, systems, or even external processes. Lastly, market risk pertains to market price volatilities and it is the relevant risk this project seeks to calibrate. A brief elaboration on market risk is provided in the next paragraph.

2.0.2 Market Risk

Market risk simply relates to the risk of losses due to the sporadic movements in the level or the volatility of market prices. [Szylar, 2013] defines it as “the risk of losses in the bank's trading book due to changes in equity prices, interest rates, credit spreads, foreign exchange rates, commodity prices and other indicators whose values are set in a public market.”. It is usually measured in two forms. Absolute risk is measured in terms of a relevant currency and focuses on the movement of total returns, while the relative risk is calculated relative to some benchmark index and focuses on the deviation from the benchmark index. Market risk can also be categorized as directional risk and non-directional risk. The former relates to the sensitivity to the direction of movements of financial variables including commodity prices, stock prices, interest rates, and exchange rates. The latter relates to the

risks from non-linear exposures and or exposures to hedged positions.

As highlighted earlier, financial risks pose serious threats to the success of financial institutions which raises the need for firms to embark on active financial risk management. Financial risk management involves the development of systems and procedures for effective diagnosis, measurement, and management of financial risks. Market risk constitutes a huge part of financial risk and the fact that the past few decades have seen increased volatility in interest rates, exchange rates, and commodity prices, as well as an influx of derivative instruments for use in the management of the risks, which makes it imperative to effectively measure and manage market risk. Several prominent experts in the literature including [Markowitz, 1952], [Garbade, 1986], and [Wilson, 1993] propose the Value-at-Risk (VaR) metric as a means to measure market risk. However, several downsides in the use of Value-at-Risk as discussed in **sections 1.8 and 1.9** below propelled experts to develop a more accurate and coherent measure. A significant number of academics in the field including [Artzner et al., 1997] and [Acerbi et al., 2001] propose a metric called Expected-Shortfall (ES) as an alternative to VaR. In the following chapters, we first discuss VaR and then ES.

3 What is VAR?

Several market risk speculation tools exist. The Greeks including delta, Vega, theta, and gamma are very technical and provide valuable information about market risk to portfolio managers but are usually of limited use to senior management. VaR is a metric that essentially compresses the Greeks and presents a concise and single-valued measure that is useful to senior management. It is perhaps one of the most common means of estimating an en-

tity's exposure to market risk owing to its simple and yet "complete" form. [Giot and Laurent, 2003] defines VaR as a measure that answers the question "With a given probability (say p), what is my predicted financial loss over a given time horizon?". [Duffie and Pan, 1997] provides a concise definition of VaR as follows: "For a given time horizon t and a confidence level p , the value at risk is the loss in market value over the time horizon t that is exceeded with probability $1-p$ ". Thus, VaR calculates the amount of losses that might accrue to an asset or a portfolio with a certain level of confidence. For instance, a one-day 95% VaR of \$2 million means that there is a 95% probability that the actual loss on the portfolio in a day will not exceed \$2 million, or that there is a 5% probability that the loss on the portfolio in a day will be worse than \$2 million.

VaR was largely developed under different motivations and using different nomenclature but it has been used widely as a means to model market risk. The following sections expatiate on the nomenclature of VaR, and the history of VaR along two lines of motivation.

3.1 History of VaR and Origins of the name Value-at-Risk

3.1.1 Origins of the name VaR

[Guldimann, 2000] implies that the name "value-at-risk" originated from JP Morgan before 1985. However, the origin of the name VAR is still not very clear. Many nomenclatures have been used to refer to the measure. The most popular names that have been used in referring to it include "dollars-at-risk" (DaR), "capital-at-risk" (CaR), "Income-at-risk" (IaR), "earnings-at-risk" (Ear), and "Value-at-risk" (VaR). [Holton, 2002] asserts that "Value-at-

risk” has been retained perhaps due to the “vagueness of the label “value” that made “value-at-risk” attractive”, but perhaps, VaR owes its popularity to the G-30 report in 1993 which was very mainstream and was the first published document to make use of the word “value-at-risk”.

3.1.2 History of VAR

- From Portfolio Theory

The earliest development of VAR sprang along the lines of portfolio theory and capital adequacy computations. In the world of portfolio theory, the first ever VAR measure to have been published was by [Leavens, 1945]. Leavens “unintentionally” identified a VAR metric when he gave a quantitative example. He proposed a ten-bond portfolio over a specific time horizon such that each bond could pay \$1000 at maturity or pay \$0 through default. According to Leavens, the events of default were independent and the value of the portfolio at maturity had a binomial distribution. Although he did not explicitly identify a VAR metric, he repeatedly mentioned “spread between probable losses and gains” hinting that he was thinking about the variance of the portfolio market value.

A few years later, [Markowitz, 1952] and [Roy, 1952] in their bid to develop a measure that would help in the selection of portfolios to optimize profits for given levels of risks, separately formulated and published very similar VAR measures. Despite the similarity in the mathematical formulation, each of them employed different VAR metrics in trying to model risk factors to reflect hedging and diversification effects. Markowitz employed a simple return metric which required only a risk factors covariance matrix constructed using Bayesian pro-

cedures. Roy used a metric of shortfall risk which required a mean vector as well as a risk factor covariance matrix. [Markowitz, 1959] described an optimization scheme guide in his book that required computational power that was nonexistent at the time. Hence, he formulated a more tractable VAR measure that used a diagonal covariance matrix. [Sharpe, 1963] described a different form of this VAR measure which motivated his Capital Asset Pricing Model (CAPM) the following year. VAR measures were proposed and published mostly in light of portfolio theory by several others including [Tobin, 1958], [Sharpe, 1963], [Lintner, 1975], and [Mossin, 1966], however, the measures they employed were mostly theoretical due to the lack of computational power in that era. The VAR measures were mainly developed to suit equity portfolios since other asset categories such as real estate, futures contracts debt instruments created modeling issues. The inception of Monte Carlo methods in VAR measures is seen in [Lietaer, 1971]. He proposed a practical VAR measure to evaluate foreign exchange risk by presenting a complex procedure for optimizing hedges since the tumultuous fixed rate regime around World War II was riddled with random devaluations.

Numerous financial events, innovations, and challenges occurred during the 1970s and 1980s: The Bretton Woods exchange agreement collapsed in 1971 allowing for floating exchange rate regimes which opened a door for a foreign exchange forward market. Also, currency swaps were introduced, [Black and Scholes, 1973] published their option-pricing model, and several assets markets including the Chicago Mercantile Exchange (CME), the Chicago Board Options Exchange (CBOE), over-the-counter (OTC) options were established. A huge consequence of

the financial innovations of this era was the proliferation of leverage with the introduction of many new instruments. As the proliferation of leverage skyrocketed, so did risk, and portfolio management sought innovative ways to manage the risk. The existing risk metrics of financial accounting were not effective and exposure metrics such as Delta, Gamma, and Vega were of limited practicality, especially to senior management. In addition, the computer age was fast developing by the 1990s and single processors could effortlessly perform very complex tasks including the procedures proposed by [Markowitz, 1959]. Financial firms exploited the technology presented by computers in the pricing of complex derivatives using Monte Carlo methods. Moreover, the players in the financial data industry such as Bloomberg and Reuters began collecting databases of historical prices providing the data required in specifying the probabilistic assumptions that would be used by VAR measures. Hence, the consequent increase in risks, the expansion in the types of assets for which VAR could be applied, and the technological and logistic advancement from the new financial tenure coupled with firms' need to measure market risks across several assets prompted a leap in the progress of VAR.

- From Capital Adequacy

Turning to the capital adequacy perspective, before 1933, most capital and securities markets were either self-regulated or were not regulated altogether. The October US stock market crash, the Great Depression, and the Paperwork Crises between 1967 and 1970 are some events that led to the loss of value in the securities market and thus necessitated the specification of capital requirements. In 1975, the US Securities and Exchange Commission (SEC) implemented the Uniform Net Capital Rule

(UNCR) to impose capital requirements on all securities firms ensuring they possessed sufficient liquidity to satisfy client obligations. Incessant interest rate volatility in the 1980s prompted the SEC to implement “haircuts” sufficient to cover losses that would be incurred during the liquidation of troubled securities with 95% confidence. Thus, the regulator essentially required securities firms to hold an amount equal to the one-month 95% VAR using a “comprehensive” approach. Before long many securities firms created different modifications of the the SEC’s VAR measure for internal risk assessment and in particular, to assess the overall market risk of the firm. [[Garbade, 1986](#)], contributed to VAR measures in the 1980s. Working in the Bankers Trust Cross Markets Research Group, he proposed complex VAR measures, assuming normally distributed portfolio market values, to compute internal capital requirements. However, his work did not gain much popularity and was mostly shared among institutional clients of Bankers Trust.

The Basle Accord in 1988 is perhaps one of the biggest implementations of a VAR measure in the late 18th century. Unlike the UNCR of the SEC of the United States of America, it was an international effort targeted at banks. The accord was established and implemented by The Basle Committee on Banking Supervision comprising representatives of central banks and regulatory authorities of Germany, Belgium, Canada, France, Italy, Japan, Luxembourg, Netherlands, Spain, Sweden, Switzerland, the United Kingdom, and the United States of America, and under the auspices of the Bank of International Settlements (BIS). Among other goals, it sought to promote uniform capital requirements to enhance competition among banks and, published a set of minimum capital requirements that aimed at protecting banks in the

the G-10 countries from credit risks. Within the European Union, the efforts in harmonizing EU markets led to the establishment of the 1993 Capital Adequacy Directive (CAD) which sought to specify uniform capital standards to banks and securities firms and was closely aligned with the Basle Accord. The CAD minimum capital requirement for market risk was based on a crude VAR measure to calculate the 10-day 95% VAR using the ‘building block’ approach. The Basle Accord was updated in the Basle-IOSCO Initiative in the early 1990s to include bank capital requirements for market risk, and these updates were also incorporated in the CAD in Europe. The Basle-IOSCO Initiative did not find much success and in April 1993, the Basle committee amended the 1988 accord which proposed minimum capital requirements for the market risk of banks. Banks were required to make their calculations based on the building-block VAR measure roughly equating to a 10-day 95% VAR metric. However, the update was generally reviewed as a step backward as the measure ignored diversification effects while available proprietary VAR measures were modeled to capture diversification effects. Hence a revision was made to the amendment which updated the building-block measure and also gave the liberty for banks to use either the updated building-block VAR measure or a regulation-approved proprietary VAR measure of their choosing to calculate capital requirements. The regulators required proprietary measures to be consistent with a 10-day 99% VAR metric.

Around the 1990s, the term “risk management” was not new but the concept was still in its infancy, and numerous non-financial firms barely practically adopted it. A report by the G-30 group titled Derivatives; Practices and Principles, and popularly referred to as the G-30 Report,

played a crucial role in financial risk management. The G-30 is a group of 30, founded in 1978 as a non-profit organization of senior executives, regulators, and academics, to deepen the understanding of international economic and financial issues. Their report mainly described derivatives while giving recommendations to dealers and end-users. The report also recommended risk be assessed using VAR and stress testing. A worthy contributor to the popularity of VAR is [[Guldimann, 2000](#)] with the development of RiskMetrics.

3.2 Mathematical Definition and Properties of VaR

VaR is an estimate of the worst possible loss on a portfolio of assets over a specific period with a given level of confidence. Thus, VaR is a function of the level of confidence and the time horizon. VaR can also be interpreted as the minimum amount of capital an investor could reserve to preserve solvency with a probability of at least α percent. Generally, if α is the stated confidence level and T is the given time horizon, then VaR is simply given by the **(100- α)th** quantile of the distribution of the changes in the value of the portfolio in the next T days [[Hull and Basu, 2016](#)]. Suppose that \mathbf{X} is a random variable representing the distribution of profits and losses on a portfolio in a given time horizon such that positive values of the distribution represent profits and negative values are losses. The mathematical description of VaR is given by the following definition.

Definition 1.1. (Quantile, VaR) Tasche (2002): Let $\alpha \in (0,1]$ be fixed and \mathbf{X} be a real random variable on the probability space $(\Omega, \Sigma, \mathbf{P})$. Define $\inf(\emptyset) = \infty$. We then call

$$q_\alpha(X) = \inf\{x \in R : P[X \leq x] \geq \alpha\}$$

the α -quantile of X . We call

$$VaR_\alpha(X) = q_\alpha(-X)$$

the VaR at (confidence) level α of X .

It is important to note that an α **T-day** VaR is approximated by an ($\alpha\%$ 1-day VaR) $\cdot\sqrt{\mathbf{T}}$. This formulation holds with equality if the distribution of the changes in the value of the portfolio over successive days are independently and identically distributed (iid), but is only an approximation in other cases. It is also worth noting that usually, the values of α that are of interest are those close to 1.

It is necessary to define a risk measure to discuss the properties of VaR. A definition of risk measure in [Tasche, 2002] is provided as follows:

Definition 1.2 (Risk Measure): let $(\Omega, \Sigma, \mathbf{P})$ be a probability space and V be a non-empty set of Σ -measurable real-valued random variables. Then any mapping $\rho: V \rightarrow R \cup \infty$ is called a risk measure. [Artzner et al., 1999]; [Kusuoka, 2001] propose the properties that a risk measure should possess. VaR satisfies some of these characteristics and these properties are explained below.

3.3 Properties of VaR

let $\alpha \in (0,1]$ be fixed and let $(\Omega, \Sigma, \mathbf{P})$ be a probability space. Consider the risk measure ρ on the set V of the Σ -measurable real-valued random variables which is given by

$$\rho(X) = VaR_\alpha(X), X \in V$$

. Then ρ has the following properties:

1. **Monotonicity:** For $X, Y \in V, X \leq Y \Rightarrow \rho(X) \geq \rho(Y)$. This means

that if a portfolio has systematically lower values in any state of the world in comparison to another portfolio in the same state of the world, then the former portfolio must have a higher risk.

2. **Positive Homogeneity:** $X \in V, h > 0, h * X \in V \Rightarrow \rho(h * X) = h * \rho(X)$. Positive homogeneity simply implies that an increase in the size of a portfolio by a positive factor should translate into a proportionate increase in the scale of the risk associated with the new portfolio.
3. **Translation Invariance:** $X \in V, a \in R, X + a \in V \Rightarrow \rho(X + a) = \rho(X) - a$. This characteristic of a risk measure implies that adding cash as an asset to a portfolio reduces the risk of the portfolio by the amount of cash in the portfolio.
4. **Law Invariance:** $X, Y \in V, P[X \leq t] = P[Y \leq t] \forall t \in R \Rightarrow \rho(X) = \rho(Y)$. This relates to the distributions of two different portfolios. It stipulates that two portfolios produce identical risk if their cumulative distributions at every interval are the same. In other words, it associates the same risk value to two portfolios with the same probability distribution for a common probability measure.
5. **Comonotonic Additivity:** for f, g non-decreasing, and for Z real random variable on $(\Omega, \Sigma, \mathbf{P})$ such that $f \circ Z, g \circ Z \in V \Rightarrow \rho(f \circ Z + g \circ Z) = \rho(f \circ Z) + \rho(g \circ Z)$. Comonotonicity refers to the positive interdependence between vectors of a random variable. Thus, comonotonic additivity implies that if portfolios have perfect dependence, then the risk of the sums of the portfolios will be equal to the risk of the composition of the portfolios. [Santos et al., 2022] describe it as "for mutually unhedgeable assets, the risk of the sum should be the sum of the risks".

The proofs of (1) to (4) are given in [Tasche, 2002], and that for (5) is given in [Denneberg, 2013].

Now that the properties of VaR have been covered, the methods of calculating VaR will be reviewed in the next section.

3.4 VaR Calculation Methods

Many techniques have been employed in the calculation of Value-at-Risk. The main methods that have been used in the estimation of VaR are the **Historical Simulation method, Variance-covariance method, and Monte Carlo Simulation method**. The major difference between these methods is the assumptions behind the distribution of the profits and losses. The Historical simulation assumes that the distribution is approximately equal to the distribution of past profits and losses, while the variance-covariance method assumes that the distribution of profits and losses is normal. Although the Monte Carlo Simulation method also assumes a probability distribution of returns, it does not impose normality on the distribution of asset returns. The following subsections describe the above-outlined methods in more detail.

3.4.1 Historical Simulation Approach

The Historical simulation method is also referred to as the **Non-Parametric Approach**. It involves the use of past data on the market factors influencing the portfolio as well as portfolio returns in the calculation of VaR in a rather practical way. The past rates of market factors and portfolio returns are used to create numerous different scenarios of the evolution of portfolio returns between the present day and the next business day. These scenarios are essentially used to construct a distribution of potential profits and losses

of the portfolio. The VaR is then estimated as the relevant percentile of the given distribution. To see this clearly, consider the illustration of the procedure in [Hull and Basu, 2016] which estimates a **one-day VaR at a 99% confidence level** using the historical simulation approach with data on \mathbf{N} relevant market factors and portfolio returns from the most recent 500 days as follows:

The methodology is illustrated in **Table 1** and **Table 2**. **Table 1** presents the observations of \mathbf{N} relevant market factors in the most recent 500 days such that the first row represents the values of the \mathbf{N} market factors on the first day the data was collected (day 0), and the second row represents the second day of the data collection and so on. Day 500 denotes the present business day. **Table 2** On the other hand shows the market variable values of the next business day (day 501) if the percentage changes between the present business day and the next business day are the same as they were between Day $(i-1)$ and Day i for $1 \leq i \leq 500$. The values of the variables on day 501 (in **Table 1**) according to the i^{th} scenario may be calculated using $v_m \frac{v_i}{v_{i-1}}$, where v_i is the value of the market variable on Day i and v_m is the recorded value of the market variable on the present day such that $m=500$.

The scenarios are outlined in such a way that the first scenario would represent the scenario in which the percentage changes in the values of all variables of the next business day are the same as they were in the first business day for which the data was recorded. The second scenario would then be the scenario such that the percentage changes in the values of all the variables of the next business day are the same as they were on the second day for which the data was collected, and the remaining scenarios would follow in the same manner. Thus, the rows of **Table 2** represent the 500 scenarios under consideration. For instance, the first row represents the

Day	Market variable 1	Market variable 2	...	Market variable N
0	20.33	0.1132	...	65.37
1	20.78	0.1159	...	64.91
2	21.44	0.1162	...	65.02
3	20.97	0.1184	...	64.90
⋮	⋮	⋮	⋮	⋮
498	25.72	0.1312	...	62.22
499	25.75	0.1323	...	61.99
500	25.85	0.1343	...	62.10

Table 1: VaR Historical Simulation Calculation Data

values of the market factors assuming that their percentage changes between the present day and the next day are the same as they were between day 0 and day 1, etc. Hence, for every given scenario, the change in the value of the portfolio between the present business day and the following business day, that is the daily return on the portfolio, is calculated. The estimated daily returns derived from the 500 different scenarios then define the distribution of the portfolio returns from which the percentile of interest is calculated. In this example, the one-day 99% VaR would be estimated by the absolute value of the first percentile of the distribution, where the first percentile is the fifth-worst daily change.

The last column of **Table 2** represents the values of the portfolio for the next business day for each of the 500 scenarios. Since the present-day value of the portfolio is known, the change in the portfolio value between the present day and the next business day is calculated for each of the 500 scenarios and then ranked. The fifth-worst change is the 1-day 99% VaR. Finally, to compute the N-Day 99% VaR for any N, it is sufficient to compute the (1-Day 99% VaR)* \sqrt{N} .

Scenario	Market variable 1	Market variable 2	...	Market variable N	Portfolio value(\$ m)
1	26.42	0.1375	...	61.66	23.71
2	26.67	0.1346	...	62.21	23.12
3	25.28	0.1368	...	61.99	22.94
⋮	⋮	⋮	⋮	⋮	⋮
499	25.88	0.1354	...	61.87	23.63
500	25.95	0.1363	...	62.21	22.87

Table 2: Generated Scenarios for Day 501 using data in Table

3.4.2 Variance-covariance Approach

This is one of the main alternatives to the Historical Simulation method and it is also known in the literature as the **Model-Building/Delta-Normal Approach**. It is sometimes referred to as the **Parametric Approach** since the main assumption surrounding this method is that the distribution of the market variables and portfolio returns is normal. Depending on the types of assets composing the portfolio, different models may be considered when discussing the Variance-covariance approach. Here, the **Linear Model** and the **Quadratic Model** presented in [Hull and Basu, 2016] will be discussed.

1. **Linear Model** Suppose that we wish to compute the **T-Day X% VaR** of a portfolio at \mathbf{P} , which is composed of \mathbf{N} assets such that α_i is invested in asset i for $1 \leq i \leq \mathbf{N}$. Let δx_i be the return on a unit of asset i in one business day. Then the change in the value of asset i in a day is $\alpha_i \delta x_i$ and the change in the value of the portfolio in one business day is given by:

$$\delta \mathbf{P} = \sum_{i=1}^{\mathbf{N}} \alpha_i \delta x_i$$

As mentioned before, the key assumption under the Variance-covariance method is that the returns on all assets are normally distributed. This implies that the equation $\delta\mathbf{P}$, being a sum of finite normal distributions is a multivariate normal distribution. Let μ_p and σ_p be the mean and standard deviation of this multivariate normal distribution. Given that the expected change in the price of a market variable over a short period is negligible, it is a customary assumption in the Model-Building Approach that the expected change in the return of an asset in a short period is approximately zero. This implies that the expected change in the value of the portfolio \mathbf{P} is also approximately zero ($\sigma_p = 0$). Now define the daily volatility of each asset by its variance σ_i^2 such that σ_i is the daily standard deviation of δx_i , and define $\rho_{i,j}$ as the correlation coefficient between the returns of the i^{th} and the j^{th} asset, that is between, δx_i and δx_j . Then the variance σ_p^2 of $\delta\mathbf{P}$ may be written as:

$$\sigma_p^2 = \sum_{i=1}^N \alpha_i \sigma_i + 2 \sum_{i=1}^N \sum_{j<i}^N \rho_{i,j} \alpha_i \alpha_j \sigma_i \sigma_j$$

The \mathbf{T} -Day standard deviation of the change in the portfolio is then $\sigma_p * \sqrt{\mathbf{T}}$.

Recall that the VaR is an estimate of the worst possible return on the portfolio with a certain degree of confidence. Hence, since the Model-Building Approach assumes that portfolio returns are normally distributed, the $\mathbf{1}$ -Day $\mathbf{X}\%$ VaR can be calculated as the $(1 - X)^{th}$ percentile of the multivariate normal distribution, and the \mathbf{T} -Day $\mathbf{X}\%$ VaR will be evaluated as $(\mathbf{1}\text{-Day } \mathbf{X}\% \text{ VaR}) * (\sqrt{\mathbf{T}})$.

Two illustrations of the Linear Model are shown below to depict scenarios where the portfolio consists of one asset and two assets respectively. The

illustrations are taken from [Hull and Basu, 2016].

- Single-Asset Case

Suppose one is interested in calculating the 10-Day 99% VaR of a portfolio only consisting of \$10 million in shares of Microsoft. Suppose μ and σ are the mean and standard deviation of the asset returns respectively. By assumption of the Variance-covariance method, the distribution of asset returns is normal and, the expected change in the asset return and thus the portfolio change over one day is equal to zero. Since the portfolio consists of a single asset, the volatility of the asset equates to the volatility of the portfolio. Suppose the daily volatility of Microsoft is 2%. Then the standard deviation of the position is equal to 2% of \$10 million or \$200,000. Suppose X is the randomly distributed asset returns, then the standard normal variable (Z) is calculated by

$$z = \frac{X - \mu}{\sigma}$$

. Since the confidence level of interest is 99%, the relevant percentile is the first percentile. The standard normal variable corresponding to the first percentile of the standard normal distribution is $Z = 2.33$. Given $\sigma = 200,000$ and $\mu = 0$,

$$2.33 = \frac{X - 0}{200,000} \Rightarrow X = 200,000 * 2.33 \Rightarrow X = 466,000$$

The One-Day 99% VaR is $X = 466,000$. This is the asset return from the normal distribution corresponding to the first percentile of the distribution. Hence, the 10-Day 99% VaR is given

by $466,000 * \sqrt{10} = \$1,473,621$.

- Two-Asset Case

Now suppose the portfolio consists of only two assets X_1 and X_2 , both normally distributed with means μ_1 and μ_2 respectively, and standard deviations σ_1 and σ_2 . respectively. Then the portfolio returns are bivariate normal and the equation from above becomes:

$$\delta P = \alpha_1 \delta x_1 + \alpha_2 \delta x_2$$

The standard deviation of the portfolio return from the equation above simplifies to:

$$\sigma_{X_1+X_2} = \sqrt{\sigma_{X_1}^2 + \sigma_{X_2}^2 + 2\rho_{X_1,X_2}\sigma_{X_1}\sigma_{X_2}}$$

As an example, suppose that the portfolio is made up of two assets, \$10 million of Microsoft shares and \$5 million of AT&T shares. Assume that the returns on the two assets follow a bivariate normal distribution with a mean of zero ($\mu_{X+Y} = 0$). Also assume that the daily volatility of Microsoft is 2% and that for AT&T is 1%, and the correlation coefficient between the two asset returns is 0.3. Let the distribution of Microsoft returns be X_1 and that for AT&T be X_2 . Then $\sigma_{X_1} = 200,000$, $\sigma_{X_2} = 50,000$ and the the standard deviation of the portfolio ($\sigma_{X_1+X_2}$) is:

$$\sigma_{X_1+X_2} = \sqrt{(200,000)^2 + (50,000)^2 + 2 * 0.3 * 200,000 * 50,000} = 220,227$$

Like the single-asset case, the One-Day 99% VaR is $2.33 * 220,227 = \$513,129$, and the 10-Day 99% VaR is $513,129 * \sqrt{10} = \$1,622,657$.

The Linear model described thus far is not very convenient for the evaluation of portfolios with positions in options. It is possible to modify the model to account for certain aspects of options.

2. Linear Model with Options

Consider a portfolio with options on a single stock whose current market price is \mathbf{S} . The delta (Δ) of the portfolio is defined as the change in the value of the portfolio for a change in the price of the stock \mathbf{S} . That is,

$$\Delta = \frac{\delta \mathbf{P}}{\delta \mathbf{S}} \Rightarrow \delta \mathbf{P} = \Delta \delta \mathbf{S}$$

Define δx as the percentage change in the price of the stock \mathbf{S} in a day such that:

$$\delta x = \frac{\delta \mathbf{S}}{\mathbf{S}}$$

, so that

$$\delta \mathbf{P} = \mathbf{S} \Delta \delta x$$

Hence, if a portfolio contains options, the change in the portfolio value in **equation 1** may be written as:

$$\delta \mathbf{P} = \sum_{i=1}^N \mathbf{S}_i \Delta_i \delta x_i$$

such that:

$$\alpha_i = \mathbf{S}_i \Delta_i$$

The standard deviation of $\delta \mathbf{P}$ can then be computed using **Equation 2** after making the relevant substitution.

Consider the following example from [[Hull and Basu, 2016](#)]. Suppose a

portfolio contains options on Microsoft with \$120 per share and AT&T with \$30 per share. Assume the options on the former have a delta of 1,000 and the options on the latter have a delta of 20,000. Using **Equation 4**, the change in the value of the portfolio is written as:

$$\delta P = 120,000\delta x_1 + 600,000\delta x_2$$

, where δx_1 and δx_2 are the daily returns on Microsoft and AT&T respectively. Assume that the daily volatility of Microsoft and AT&T are 2% and 1% respectively, with a correlation coefficient of 0.3, then the standard deviation of the portfolio change can be derived from **Equation 2** as:

$$\sqrt{(120 * 0.02)^2 + (600 * 0.01)^2 + 2 * 0.3 * 120 * 0.02 * 600 * 0.01} = 7,099$$

Since $N(-2.33) = 0.01$, The One-Day 99% VaR is $2.33 * 7,099$, and the 10-Day 99% VaR is

$$2.33 * 7.099 * \sqrt{10} = \$52,306$$

Although this modified formulation of the Linear Model does a better evaluation of portfolios with options compared to the original linear model, it fails to factor in key variables of options. In particular, it does not account for the Gamma of the options in the framework. The framework presented in the next subsection describes a model under the Variance-covariance Approach for evaluating the VaR of a portfolio with options taking into account the Gamma of the options.

3. The Quadratic Model

The Gamma holds essential information about options. [[Guidi and Pauletti, 2008](#)]

describes Gamma as "a measure of the rate of change of delta with respect to changes in the price of the underlying asset". It represents the curvature of the relationship between the value of the portfolio and the price of the underlying asset. The Gamma is very important for its impact on the distribution of the portfolio value. A positive Gamma results in a positively skewed (less fat left tail) distribution of the portfolio value and a negative Gamma results in a negatively skewed distribution (fatter left tail). Recall that VaR estimation is largely dependent on the left tail of the distribution. As a consequence, ignoring the Gamma of options in the evaluation of a portfolio will tend to overestimate or underestimate the VaR since the main assumption under the model-building Approach is that the distribution of the portfolio is normal. In particular, if a portfolio contains positions in options such that the Gamma of the portfolio is positive, assuming a normal distribution for the portfolio will overestimate the VaR and vice versa.

Hence, for a more accurate estimation of VaR, the **Quadratic Model** employs both the Delta and the Gamma to relate the distribution of the portfolio to the distribution of the asset returns. The **quadratic Model** is described as follows: Consider a portfolio with a single asset whose price is \mathbf{S} and suppose that Delta and Gamma of the portfolios are Δ and Γ respectively. The delta-neutral representation of a portfolio derived from a Taylor Series expansion of the change in the value of a portfolio over a short time horizon in [Hull and Basu, 2016] is the following equation:

$$\delta\mathbf{P} = \Delta\delta\mathbf{S} + \frac{1}{2}\Gamma(\delta\mathbf{S})^2$$

This includes the gamma of the portfolio and is thus a better approximation of portfolio value distribution compared to the linear model for options. De-

fine δx as the percentage change in the price of the stock \mathbf{S} in a day such that:

$$\delta x = \frac{\delta \mathbf{S}}{\mathbf{S}}$$

then the delta-neutral equation becomes:

$$\delta \mathbf{P} = \mathbf{S} \Delta \delta x + \frac{1}{2} \mathbf{S}^2 \Gamma (\delta x)^2$$

Given the assumption that the δx_i 's are normally distributed with a mean μ_{x_i} and standard deviation σ_{x_i} , notice that $\delta \mathbf{P}$ is not normal being a quadratic of a normally distributed variable. However, the first three moments of $\delta \mathbf{P}$ can be written as follows:

$$E(\delta \mathbf{P}) = \frac{1}{2} \mathbf{S}^2 \Gamma \sigma^2$$

$$E[(\delta \mathbf{P})^2] = \mathbf{S}^2 \Delta^2 \sigma^2 + \frac{3}{4} \mathbf{S}^4 \Gamma^2 \sigma^4$$

$$E[(\delta \mathbf{P})^3] = \frac{9}{2} \mathbf{S}^4 \Delta^2 \Gamma \sigma^2 + \frac{15}{8} \mathbf{S}^6 \Gamma^3 \sigma^6$$

The first and the second moments of the distribution can be fitted to a normal distribution. Define the mean and the standard deviation of $\delta \mathbf{P}$ as μ_p and σ_p respectively such that

$$\mu_p = E(\delta \mathbf{P})$$

and

$$\sigma_p^2 = E[(\delta \mathbf{P})^2] - [E(\delta \mathbf{P})]^2$$

The skewness (ξ_p) of the distribution of \mathbf{P} can be written as:

$$\xi_p = \frac{1}{\sigma_p^3} E[(\delta \mathbf{P} - \mu_p)^3] = \frac{E[(\delta \mathbf{P})^3] - 3E[(\delta \mathbf{P})^2]\mu_p + 2\mu_p^3}{\sigma_p^3}$$

The q^{th} percentile of the distribution of $\delta\mathbf{P}$ can then be estimated using the **Cornish-Fisher Expansion** as

$$\mu_p + \omega_q \sigma_p$$

where

$$\omega_q = z_q + \frac{1}{6}(z_q^2 - 1)\xi_p$$

, such that z_q is the q^{th} percentile of the standard normal distribution $\phi(0, 1)$. Consider an example from [Hull and Basu, 2016]. Suppose that the daily mean and standard deviation of a certain portfolio are respectively estimated as $\mu_p = -0.2$ and $\sigma_p = 2.2$, and suppose that the portfolio value distribution is negatively skewed with skewness of $\xi = -0.4$. Assuming that the distribution of $\delta\mathbf{P}$ is normal, the first percentile of the distribution will be

$$-2.33 * 2.2 - 0.2 = -5.326$$

This implies a One-Day 99% VaR of -5.326.

On the other hand, using the Quadratic model and employing the Cornish-Fisher expansion to adjust for skewness assuming a 99% confidence level, ie, $z_{0.01} = -2.33$,

$$w_q = -2.33 - \frac{1}{6}((-2.33)^2 - 1) * 0.4 = -2.625$$

The One-Day 99% VaR defined by the first percentile of the portfolio value distribution is given by

$$-0.2 + -2.625 * 2.2 = -5.976$$

Notice that assuming $\delta\mathbf{P}$ is normal underestimates the VaR.

3.4.3 Monte Carlo Simulation Approach

Also referred to as the **Semi-Parametric Approach**, the Monte Carlo approach is one of the popular methods of calculating VaR. It shares some similarities with both the Historical Simulation and the Variance-covariance approach. In particular, like the Variance-covariance method, it requires an assumption about the distribution of market variables. The distribution of choice is usually believed to approximately capture the potential changes in the market variables. Like the Historical Simulation method, it generates a multitude of scenarios about the change in the value of the portfolio with the main difference being that with the Monte Carlo method, randomly generated data from the chosen distribution are used instead of actual historical data. The advantage here is the liberty of choosing any distribution that is deemed adequately representing the changes in market factors, as well as the fact that more scenarios can be generated given the possibility of generating a limitless number of pseudo-random numbers. However, a disadvantage of this approach is that it requires a lot of time in the computation of the VaR.

Step I [[Linsmeier and Pearson, 2000](#)] describes the steps that may be employed in the computation of VaR using the Monte Carlo approach.

STEP I: The first step involves the identification of the market variables affecting the portfolio, and expressing the mark-to-market value of the portfolio in terms of the market variables.

STEP II: The next step involves the specification of a distribution that adequately resembles the actual distribution of the changes in the market variables, and the estimation of the parameters of the distribution. Unlike the other approaches in which the distribution of market variables is implic-

itly assumed in the method, this step grants some degree of freedom to the researcher concerning the choosing of the distribution for the potential future changes in the market factors.

STEP III: In this step, \mathbf{R} hypothetical values of the change in market variables are generated from the distribution selected in the previous step using a pseudo-random number generator. Since larger sample sizes are generally more ideal, \mathbf{R} is usually a very large number. The generated values of the market factors are then used to construct \mathbf{R} different scenarios for mark-to-market values of the portfolio profits and losses.

STEP IV: In the last step, the VaR is estimated in the fashion of the Historical Simulation approach. That is, the different mark-to-market profits and losses are ordered from gravest losses to biggest profits, and then the VaR is calculated as the relevant percentile of the distribution of the portfolio profits and losses.

3.5 Stress Testing and Back Testing

The world economy has been gravely affected by many events that saw huge repercussions in various economic and financial sectors. The US stock market crash in October 1929 and the Great Depression led to the stock market losing 20% of its value. The COVID-19 pandemic in 2020 also created substantial economic and financial crises around the globe. These events prompted negative ripple effects in the financial sector and other sectors worldwide. Although VaR measures the risk associated with the scarce tail ends of events in the financial sector, it is important to note that an implicit assumption in the computation of VaR is that the market is "regular" and not affected by such extreme events. Hence, for companies to provide a complete assessment of the risk profile of their portfolios, they usually employ an additional

assessment tool termed **Stress Testing**.

[[Kupiec, 2002](#)] describes Stress Testing as a method that involves the quantification of the magnitude of losses that may accrue to a portfolio under scenarios that are less likely to occur than those considered in the calculation of VaR. Stress testing essentially involves the estimation of the performance of a portfolio had it been subjected to the extreme of scenarios or events. [[Dupačová and Polivka, 2007](#)] states that the events that may be considered as scenarios in stress testing stem from historical experience—*Historical Stress Testing*, or scenarios based on speculation given current changes in macroeconomic, socioeconomic, or political factors—*Prospective Stress Testing*. Thus, Stress testing takes into account extreme scenarios that have intermittently happened in the past or are likely to happen in the future albeit with a slim chance that is virtually not possible according to the assumptions placed on the probability distributions of market variables. For instance, to evaluate the performance of a portfolio as a result of the impact of an extreme event, the percentage changes of the market variable affecting the portfolio may be set to values that were realized during a certain extreme event in the past.

Irrespective of the method used in the calculation of VaR, it only seems reasonable to employ some testing procedure to guarantee the authenticity of the method. A tool mostly used by companies to evaluate the performance of their VaR estimation approach is **Back Testing**. [[Campbell, 2005](#)] defines Back Testing as a way of gauging the accuracy of a putative VaR measure. He also asserts that while Back Testing procedures may be different with the details, the majority of the methods focus on particular transformations of the VaR and realized profits or losses. Back Testing provides an essential reality check by considering how the VaR estimates would perform in the past.

[Christoffersen, 1998] proposed a procedure for determining the accuracy of a VaR model. In this procedure, he concludes that the accuracy of a VaR model can be determined provided a sequence of *indicator values* (values from an indicator function), which are dependent on the VaR over a fixed time horizon satisfies two properties; **Unconditional Coverage Property** and **Independence Property**. In general, the idea behind Back Testing is the following: Suppose that the firm is interested in calculating the 1-Day 99% VaR, Back testing aims to evaluate the frequency with which the loss in a day is greater than the 1-Day 99% VaR estimated for that day. If the losses exceeded the VaR estimate at most 1% of the days, then the VaR methodology is considered sound.

3.6 Disadvantages of VaR Computation Methods

The VaR estimation methods that have been discussed in the previous subsection are relatively straightforward in terms of the methodology. However, every one of the methods discussed has some setbacks that inhibit them from completely representing reality. This section details the disadvantages of each of the VaR computation approaches already described.

3.6.1 Variance-covariance Approach

The Variance-covariance method is perhaps the simplest approach for computing VaR. It does not make use of historical data on market variables and it does not require the use of complex computation methods applied in the Monte Carlo methods. It simply assumes that market factors and portfolio returns follow a normal distribution. It turns out that the main setback to this method is the underlying assumption of normality. In fact, for many financial assets, historical data on standardized portfolio returns are found

to follow a student **t-distribution** with 5 degrees of freedom [Sollis, 2009]. This implies that empirical data suggests that the probability distribution of many financial assets has fatter tails relative to the tails of the standard normal distribution assumed in the Variance-covariance methodology. A consequence of this misspecification is an inaccurate VaR measure.

3.6.2 Historical Simulation Approach

The upside to the Historical simulation method is that it does not make any assumptions regarding the distribution of market variables or portfolio returns. As such, it might present a better methodology than the Variance-covariance approach in the case where the actual probability distribution is non-normal. The main issue with the Historical Simulation method is the sensitivity of the VaR estimates to the changes in the size of the historical samples used. This means that a particular choice of the historical sample size can reflect much less risk in the VaR estimate than there is, and conversely, a different sample size choice can reflect more risk than there needs to be. In addition, the sensitivity issue of the Historical Simulation method makes VaR estimates tend to be pro-cyclical because in times of positive market outlook VaR estimates tend to be low. This could be problematic since minimum requirements based on VaR estimates would allow less minimum capital holdings and thus increase risk in case of bubbles. [Sollis, 2009] shows that smaller sample periods cover bull markets with low volatility compared to larger volatility associated with extensive periods that have huge negative returns due to extreme financial events such as Black Monday of October 1987 and the dotcom crash of March 2000.

3.6.3 Monte Carlo Simulation Approach

This methodology also employs a specified probability distribution that the researcher believes is adequately representative of the actual distribution of the change in market variables and portfolio returns. It is possible that the assumed distribution widely differs from the true distribution in which case the VaR estimate would likely undermine the risk associated with the portfolio. For instance, if the true distribution of portfolio returns is a student **t-distribution** with 5 degrees of freedom but the researcher also assumes a student **t-distribution** with 1 degree of freedom, the **t-distribution** with one degree of freedom has significantly more fat tails compared to the one with 5 degrees of freedom. This means that the VaR estimate would be over-estimated.

3.7 The Problem with VaR: Coherence and Subadditivity

It is clear from the previous subsection that the methods used in the computation of VaR have some limitations and are thus far from perfect. The results from implementing different methodologies or even the same methodologies mostly produce VaR results that are inconsistent. [Beder, 1995] calculated sixteen VaR estimates for three different portfolios using the Historical Simulation method and the Monte Carlo Simulation method and found that all the sixteen measurements were inconsistent. [Marshall and Siegel, 1996] also estimated the One-Day 95% VaR of the same portfolios and the same covariance matrices through eleven VaR software vendors and found that the vendors produced different results. Furthermore, [Taleb, 1997] affirms that "VaR measures are conceptually flawed".

Perhaps the most significant and most popular challenge with VaR as a risk measure is that it does not qualify as a *coherent* risk measure. [Artzner et al., 1999] define a general and complete set of axioms that a sound risk measure must satisfy. A risk measure satisfying all the axioms is *coherent*. In particular, if a portfolio is riskier than another, it will always have a greater risk value provided that the risk measure is *coherent*, [Acerbi et al., 2001]. On the other hand, a risk measure that does not meet all the axioms of a risk measure may produce somewhat paradoxical and misleading results. The axioms proposed by [Artzner et al., 1999] that a *coherent* risk measure must satisfy are **Translation Invariance**, **Positive Homogeneity**, **Monotonicity**, and **Subadditivity**. Recall that the first three of these axioms have been discussed in **Section 1.5 (Properties of VaR)**. The VaR measure is not considered a *coherent* risk measure in general since it fails to satisfy the Subadditivity axiom. The VaR measure meets the subadditive requirement only in the case where all market variables and portfolio returns are assumed normal. However, the real world is not always characterized by the Gaussian world.

Subadditivity emphasizes the significance of diversification in a portfolio. It implies that diversifying a portfolio does not create any additional risk. In other words, according to this property, the total risk from the combination of two portfolios should be less or equal to the sum of the individual risk of the portfolios. Concerning banking and other financial supervisory and regulatory framework, subadditivity implies that the total risk of a bank or other financial institutions should be at most equal to the risk accumulated from each of the branches. In another sense, if each of the branches of a financial institution satisfies the capital adequacy requirements, then the financial institution as a whole should be adequate in terms of the capital

requirements.

Using the denotations in **section 1.5**, subadditivity is mathematically defined by [[Artzner et al., 1999](#)]; [[Kusuoka, 2001](#)] as follows: **Subadditivity**: For all X and $Y \in V$, $\rho(X + Y) \leq \rho(X) + \rho(Y)$.

Another critique of the VaR measure is that, for instance, although it estimates the loss on a portfolio that cannot be exceeded 99% of the time, it does not specify just how much loss can be suffered the other 1% of the time. A coherent risk measure that can remedy this limitation of VaR is **Expected-Shortfall**. The next section takes a deep dive into this coherent risk measure.

4 Expected-Shortfall (ES)

Expected-Shortfall sometimes referred to as **Conditional Value-at-risk(CVaR)** is a risk measure that presents itself as a solution to circumvent most of the limitations of VaR. It qualifies as a coherent risk measure since it satisfies all the axioms of coherence, and it estimates the extent of loss that can be accrued on a portfolio in the remaining rare event. Hence, ES comes off as a natural alternative to VaR for financial risk management. The following subsections detail the motivation for the development of ES and the methodology for the calculation of ES.

4.1 Motivation and Building Blocks

Many researchers including [Beder, 1995] and [Marshall and Siegel, 1996] have in the past raised concerns about the consistency and soundness of the Value-at-Risk measure. Despite these concerns, a vast majority of banks and regulators adopted VaR as the best practice and conventional method of assessing the market risk of portfolios and establishing minimum capital requirements. The axioms formulated in *Thinking Coherently* by [Artzner et al., 1997] which was followed by *Coherent Measures of Risk*, [Artzner et al., 1999] created a science of risk management based on a deductive axiomatic framework. Thus, making use of measures of risk not consistent with this science of risk management would mean using the wrong tool for the measurement of risk in general. [Acerbi and Tasche, 2002a] emphasized this point by saying:

”Writing axioms means crystallizing, in a minimal number of precise statements, the intrinsic nature of a concept. It is a necessary step to take in the process of translating a complex reality into a mathematical formulation.

The axioms of coherence simply embody synthetically and essentially those features that single out a risk measure in the class of statistics of portfolio dynamics, just like the axiom ‘it must be higher when air is hotter’ identifies a measure of temperature out of the class of thermo-dynamical properties of the atmosphere. If you want to use a barometer for measuring temperature although pressure does not satisfy the above axiom, you need not be surprised if you happen to be dressed like an Eskimo on a hot cloudy day or to be wearing a swimming costume in icy sunshine”.

The insurgence of the axiomatic appeal of risk measurement drew the attention of risk analysts, risk practitioners, and researchers to the realization that VaR fails to conform to the basic rules and regulations that are defined by the framework created through the proposed science of risk management. Many researchers and practitioners express their disregard for the VaR measure as a risk measure for the reason that it does not belong to the family of coherent risk measures.

Although the properties of a coherent risk measure were defined in 1999, a measure satisfying all the axioms to replace VaR did not emerge until after a few years. The term ES originates from [Acerbi et al., 2001], however, was first mentioned by [Delbaen, 2002] and was first independently proposed as CVaR by [Rockafellar and Uryasev, 2002]. The next subsection provides the definition and properties of ES.

4.2 Definition and Properties of Expected-Shortfall

Consider the definition of ES in [Tasche, 2002] as follows:

Definition 2.1 (Expected-Shortfall): Let $\alpha \in (0, 1)$ be fixed and \mathbf{X} be a real random variable on a probability space $(\Omega, \Sigma, \mathbf{P})$ with $E[\max(0, -\mathbf{X})] <$

∞ . Define $q_\alpha(-\mathbf{X})$ as in **Definition 1.1. (Quantile, VaR)**. The following:

$$\mathbf{ES}_\alpha(\mathbf{X}) = -(1 - \alpha)^{-1}(E[\mathbf{X}\mathbf{I}_{\{-X \geq q_\alpha(-X)\}}] + q_\alpha(-\mathbf{X})\alpha - P[-\mathbf{X} < q_\alpha(-\mathbf{X})]),$$

where $\mathbf{I}_\mathbf{A} = \mathbf{I}_\mathbf{A}(\mathbf{a})$ is the indicator function of the set \mathbf{A} such that $\mathbf{I}_\mathbf{A}(\mathbf{a}) = 0$ if $\mathbf{a} \notin \mathbf{A}$ and $\mathbf{I}_\mathbf{A}(\mathbf{a}) = 1$ if $\mathbf{a} \in \mathbf{A}$, and $\mathbf{ES}_\alpha(\mathbf{X})$ is the Expected-Shortfall at level α of \mathbf{X} .

The literal interpretation of the mathematical description of ES in **Definition 2.1** is "the average loss in the worst $100\alpha\%$ cases". Framed differently, ES at an $\alpha\%$ level measures the expected return on a portfolio in the worst $\alpha\%$ of cases. Unlike VaR, ES is an average; it measures the mean loss on the portfolio such that the loss occurs at or below the q -quantile. As such, ES is also known in the literature as **Average Value-at-Risk (AvaR)**.

4.2.1 Properties of ES

The most relevant property of ES is *coherence*. As mentioned earlier, ES is a coherent risk measure, meaning that it satisfies the axioms of **Monotonicity, Positive Homogeneity, Translation Invariance, and Subadditivity**. These axioms have already been discussed in **section 1.5** and **section 1.9**. Aside from Coherence, other useful properties of ES are **Integral Representation, Continuity, and Monotonicity in the level of confidence α** . They are discussed below.

Integral Representation: ES can be represented as an integral as follows.

Definition 2.2 If X is a real-valued random variable on a probability

space $(\Omega, \Sigma, \mathbf{P})$ with $E[X^{-1}] < \infty$ and $\alpha \in (0, 1)$ is fixed, then

$$\bar{x}_{(\alpha)} = \alpha^{-1} \int_0^\infty x_{(u)} du$$

where $x_{(u)}$ is the lower u -quantile of X , and $\bar{x}_{(\alpha)}$ is as defined as $-\mathbf{ES}_\alpha(X)$.

Continuity in the level of Confidence (α): Financial companies often have to deal with random variables with discontinuous probability distributions. For instance, non-traded loans have purely discrete distributions, and derivatives mostly have a mixture of discrete and continuous distributions. One issue with measures estimated from the tail ends of a distribution such as VaR when applied to discontinuous distributions is their sensitivity to the level of confidence [[Acerbi and Tasche, 2002b](#)]. The implication of this is that such measures are not generally continuous with respect to the level of α . ES has the advantage of being continuous in the level of confidence, with the pleasant consequence that irrespective of the nature of the underlying distributions, the risk estimated by ES will not be dramatically affected when the level of confidence is slightly changed. Mathematically, continuity in the α level is defined in **Corollary 2.3** below:

Corollary 2.2: If X is a real-valued random variable with $E[X^{-1}] < \infty$, then the mappings $\alpha \mapsto \bar{x}_\alpha$ and $\alpha \mapsto \mathbf{ES}_\alpha$ are continuous on $(0,1)$.

Monotonicity in the level of Confidence (α): Monotonicity in the level of α is transcended as a natural phenomenon of any sound risk measure. It simply implies that the smaller the specified level of α the greater the risk. Mathematically, it is defined in the proposition that follows.

Proposition 2.4: If X is a real-valued random variable with $E[X^{-1}] < \infty$, then for any α in $(0,1)$ and any $\epsilon > 0$ with $1 + \epsilon < 1$ we have the following inequalities:

$$\bar{x}_{(\alpha+\epsilon)} \geq \bar{x}_{(\alpha)}$$

and

$$\mathbf{ES}_{(\alpha+\epsilon)} \leq \mathbf{ES}_{(\alpha)}.$$

The proofs for the properties of ES are done by **Acerbi and Tasche (2002)**.

4.3 Expected-Shortfall Estimation Methods

Numerous methods exist for the computation of ES. The majority of these methods can be categorized under **Parametric methods**, **Non-parametric methods**, and **Semi-parametric methods**. [[Nadarajah et al., 2014](#)] provide a review of many of the methods that have been developed in recent years. This section discusses some of these methods in each category according to [[Nadarajah et al., 2014](#)].

4.3.1 Parametric Methods

The methods of estimating ES that are considered here are those that are employed in cases where data from parametric distributions are used and where parameters are specified. The list of parametric ES calculation methods is long. They are mostly based on the following: Gaussian distribution, Johnson family method, Azzalini's skewed normal distribution, Azzalini's skewed normal mixture distribution, Student's t distribution, Azzalini, and Capitanio's skewed t distribution, Jones and Faddy's skewed t distribution, generalized asymmetric t distribution, non-central t distribution, stable distribution, generalized hyperbolic distribution, normal mixture distribution, stable mixture distribution, Students' t mixture distribution, generalized Pareto distribution, asymmetric exponential power distribution, generalized asymmetric Students' t distribution, Mittnik and Paolella's generalized t distri-

bution, asymmetric Laplace distribution, elliptical distribution, multivariate gamma distribution, random walks, autoregressive process, GARCH (1, 1) process, quantile regression method, location-scale distributions, RiskMetrics model, QGARCH (1, 1) model, QGARCH (p, q) model, and the block minimum method.

A few of the most popular parametric methods are discussed as follows:

1. Gaussian Distribution

This is the most popular among the set of parametric methods. It is defined as follows: Suppose that the underlying distributions follow a Gaussian distribution with mean μ and variance σ^2 such that X_1, X_2, \dots, X_n are the Gaussian observations, Then the ES is computed as:

$$\widehat{\mathbf{ES}}_\alpha = E[X|X > s\Phi^{-1}(a)],$$

where s is the sample standard deviation given by:

$$s = \sqrt{\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2},$$

and \bar{X} is the sample mean.

The major challenge of this model is that it is unable to model skewed data. Hence, models based on **Azzalini's Skewed normal distribution** and **Azzalini's Skewed normal mixture distribution** in [Azzalini, 1985] and [Bernardi, 2013] respectively, which are variants of the Gaussian model, endeavor to remedy this weakness. Other extensions that aim at improving the flexibility of the Gaussian model are based on **normal mixture distribution** and **Stable mixture distribution** in [Broda and Paolella, 2011].

2. Student's t-distribution

With respect to empirical financial data, the tails of the student's t-distribution are more realistic compared to those of the Gaussian case. This method is defined as follows: Let X denote a variable from the student t-distribution with scale parameter $c \geq 0$, location parameter $-\infty < \mu < \infty$, and $n \geq 0$ degrees of freedom. Let $f_X(x)$ be the probability density function of X and let q_p denote the p^{th} quantile of the standard t-distribution, that is $\Pr(X \leq q_p) = p$ when $\mu = 0$ and $\sigma = 1$. Then [Broda and Paoletta, 2011] estimates ES as:

$$\mathbf{ES}_p(X) = \frac{1}{p} Ttail(q_p, n),$$

where

$$Ttail(c, n) = -\frac{n^{-1/2}}{\sigma B(n/2, 1/2)} \left(1 + \frac{c^2}{n}\right)^{-(n+1)/2} \left(\frac{n + c^2}{n - 1}\right).$$

As with the Gaussian case, a weakness of the student's t-distribution is its inability to model skewness. The variants of this model that attempt to remedy this weakness include the models based on **Azzalini and Capitanio's skewed t-distribution**, **Jones and Faddy's skewed t-distribution**, **Generalized asymmetric t-distribution**, and **Non-central t-distribution** by [Broda and Paoletta, 2011]. Another variant is the **Generalized asymmetric Students' t-distribution** in [Zhu and Galbraith, 2010]. An extension of the ES calculation method based on the students' t-distribution that aims at improving flexibility is the **Students' t-mixture distribution** in [Broda and Paoletta, 2011].

3. Asymmetric Laplace distribution

Another useful method for calculating ES is by using the asymmetric Laplace distribution. [Chen et al., 2012] show using empirical data that this method is superior to other estimators of ES that are based on normal and student t-distributions among others. The probability density function of the asymmetric Laplace distribution as defined by [Lu et al., 2010] is :

$$f(x) = \frac{b}{\tau} \exp\left[-\frac{b}{\tau}|x - y|\left(\frac{1}{c}I(x < y) + \frac{1}{1-c}I(x > y)\right)\right],$$

where $b = \sqrt{c^2 + (1-c)^2}$, γ is the location parameter, τ is the scale parameter, and c is the shape parameter. Then the ES is defined as the following:

$$\mathbf{ES}_p = \frac{c}{b} \left[\log\left(\frac{p}{c}\right) - 1 \right],$$

for $0 \leq p < c$.

4. Random walks

The evolution of most financial variables is widely modeled as random walks. Thus, [Embrechts et al., 2005] propose an ES estimator based on a random walk as follows: Let X_t denote financial returns and suppose X_t be a random walk such that $X_t = X_{t-h} + r_t$, r_t are normal and independent random variables with mean μ and variance σ^2 . Suppose

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^n r_{ih}, \hat{\sigma} = \sqrt{\frac{1}{n-1} \sum_{i=1}^n (r_{ih} - \hat{\mu})^2} : \hat{\sigma}^k = \sqrt{k} \hat{\sigma}, \hat{\mu}^k = k \hat{\mu}.$$

Then the ES estimator is given by:

$$\widehat{\mathbf{ES}}_p = \frac{1}{p} \exp(\hat{\mu}^k + \frac{(\hat{\sigma}^k)^2}{2}) \Phi(\Phi^{-1}(p - \hat{\sigma}^k) - 1),$$

where $\Phi(\cdot)$ denotes the standard normal cumulative distribution function. Natural extensions of this model are based on the **Autoregressive process** and the **GARCH(1,1) process** by [Embretchts et al., 2005]. Other extensions include the **QGARCH (1,1) model** and the **QGARCH (p,q) model** by [So and Wong, 2012].

4.3.2 Non-parametric Methods

The methods of estimating ES that are considered here are those that are employed in cases where data are not assumed to follow any particular parametric distribution. There are many such non-parametric methods for calculating ES. The most popular ones are the historical method, filtered historical method, Brazauskas et al.'s estimator, Yamai and Yoshiba's estimator, Inui and Kijima's estimator, Chen's estimator, Peracchi and Tanase's estimator, Jad-hav et al.'s estimators, kernel method, trimmed kernel method, and Richardson's method.

1. Historical Method

This method is widely considered the best method for the computation of ES. The definition is provided below. Suppose $X_{(1)} \leq \dots \leq X_{(n)}$ is the order statistics from lowest to highest that correspond to the returns X_1, \dots, X_n . Then the ES is estimated as follows:

$$\widehat{ES}_p(X) = \left(\sum_{i=[np]}^n X_{(i)} \right) / (n - [np]).$$

A variant of the Historical method is the **Filtered Historical method** by [Magadia, 2011], where returns are modeled based on the ARMA-GARCH model. Another modification to the Historical method is the model by [Jadhav et al., 2009]. They propose two ES estimators, and they illustrate that both estimators are better than the estimator based on the Historical method when using data on several stock indices.

2. Kernel Method

Another popular method of computing ES is the kernel method. It is described as follows: Suppose $X_{(1) \leq \dots \leq X_{(n)}}$ represents the order statistics in ascending order which correspond to the returns X_1, \dots, X_n . Let $K(\cdot)$ be a symmetric kernel and let h denote a suitable bandwidth such that $K_h(u) = (1/h)K(u/h)$, $A(x) = \int_{-\infty}^x K(u)du$ and $A_h(u) = A(u/h)$. Then the ES as proposed by [Yu et al., 2010] is defined as:

$$\widehat{ES}_p(X) = \frac{1}{np} \sum_{i=1}^n X_i A_h(\widehat{q}(p) - X_i),$$

where

$$\widehat{q}(p) = \sum_{i=1}^n \left[\int_{i-1/n}^{i/n} K_h(t-p) dt \right] X_{(i)}.$$

A modification of this method is the **Trimmed Kernel Method** by [Hill and Renault, 2012].

3. Yamai and Yoshiba's Method

[Yamai and Yoshiba, 2002] proposed one of the earliest methods for estimating ES. They define their ES estimator as follows: Suppose $X_{(1) \leq \dots \leq X_{(n)}}$ is the order statistics from lowest to highest that correspond

to the returns X_1, \dots, X_n . Then the ES is estimated as below:

$$\widehat{ES}_p(X) = \frac{1}{n(\alpha - \beta)} \sum_{i=n\beta}^{n\alpha} X_{(i)},$$

assuming α is much larger than β . [Chen, 2008] also proposes a similar ES estimator.

4.3.3 Semi-parametric Methods

Semi-parametric methods are a mix of parametric and non-parametric methods. The ES methods discussed here possess elements from both parametric and non-parametric methods. The estimator in [Necir et al., 2010] and a method based on **Heavy-tailed processes** are discussed here.

1. Necir et al.'s Estimator

The ES estimation method proposed by [Necir et al., 2010] is defined as follows: Suppose $X_{(1)}, \dots, X_{(n)}$ denotes the order statistics from lowest to highest corresponding to the returns X_1, \dots, X_n . Then the ES is estimated by:

$$\widehat{ES}_p = \frac{1}{p} \int_{k/n}^p \widehat{F}^{-1}(t) dt + \frac{kX_{(n-k)}}{np(1 - \widehat{\gamma})},$$

where

$$\widehat{\gamma} = \frac{1}{k} \sum_{i=1}^k \log \frac{X_{(n-i+1)}}{X_{(n-k)}}$$

and $\widehat{F}^{-1}(\cdot)$.

2. Heavy-Tailed Process

Suppose that the sequence of returns is a heavy-tailed process; that is $r_t = X_t = X_{t-h}$ satisfies $\Pr(r_t < -x) \propto x^{-a}L(x)$ as $x \rightarrow \infty$, where

$\alpha > 0$ and $L(\cdot)$ is a slowly varying function. [Embrechts et al., 2005] estimate the ES as:

$$\widehat{ES}_p = \frac{1}{p} \int_0^p \exp\left[\left(\frac{kl_{n,p}}{nq}\right)^{1/\widehat{\alpha}_{l,n,p,n}r_{l,n,p}}\right] dq^{-1},$$

where

$$\widehat{\alpha}_{l,n} = \left[\frac{1}{l} \sum_{i=1}^l \log\left(\frac{r_{(i)}}{r_{(l)}}\right)\right]^{-1}$$

is the **Hill** estimator of α .

Although several methods for estimating ES from the different categories exist, with some methods being superior to others, it is indisputable that ES methods have some flaws in a general sense. The next subsection outlines some of the weaknesses of ES.

4.4 Downsides of Expected-Shortfall

[Delbaen, 2002], [Acerbi and Tasche, 2002b], and [Kusuoka, 2001] show that ES strikes as a preferable measure of market risk compared to VaR. In particular, they show that ES is the smallest coherent and law-invariant "majorant" of VaR. Although ES was introduced as a major upgrade to VaR and as a tool to circumvent the challenges of VaR, it has its shortcomings.

Firstly, a desirable property of a risk measure is that increased risk should be penalized by an increased measure. Increased risk can be modeled by the degree-three convex order. Penalization implies that measures preserve higher degree convex orders and such measures are termed as tail-preserving distortion measures. However, [Hürlimann, 2004] shows that ES does not preserve some higher degree stop-loss orders. Specifically, he shows that ES measure as measured by the convex third order degree is not consistent with

right tail risk.

Another downside of ES is that relative to VaR it requires a relatively larger sample data size to produce results of the same level of accuracy. [Yamai and Yoshida, 2002] show that if the underlying distributions are fat-tailed, ES estimation errors are much larger compared to those of VaR. In this case, reducing the estimation errors requires a much larger sample size.

4.5 Some Applications of Value-at-Risk and Expected-Shortfall

Expected-Shortfall and Value-at-Risk have been applied extensively over recent years. They have been applied in fields including banking, insurance, securities and exchange, energy, and even in the movie industry. In banking and insurance, [Gao and Li, 2009] use ES, VaR, and other estimation methods to compare the annual operational capital of listed and non-listed banks in China. [Lee and Fang, 2010] also used VaR and ES for the analysis of operational losses and operational risk capital in commercial banks in Taiwan.

In securities and currency exchange, [Rachev et al., 2007] use ES to evaluate momentum strategies based on reward–risk stock selection criteria versus ordinary momentum strategies based on cumulative return criteria using 517 S&P 500 stocks from 1996 to 2003. [Xiu-Min and Fa-Chao, 2006] also employ VaR and ES based on the Extreme Value Theorem (EVT) to perform an empirical analysis of Equity Risk for the Shanghai Stock Market. Moreover, [YU and TAO, 2008] assess the share of risk by different industries on the Chinese stock market. In addition, [Krehbiel and Adkins, 2008] apply both risk measures to evaluate daily changes in U.S. dollar London

inter-bank offer rates, and [Wang et al., 2010] apply them to assess the risk of exchange of the Chinese Yuan.

With regards to energy and the environment, [Lindström and Regland, 2012] employ VaR and ES in modeling the extreme interdependence between electricity markets in Europe. [Sims and Kamal, 1996] also apply the concepts of repowering coastal stations in a bid to augment water supplies in southern California.

VaR and ES are also applied in the movie industry by [Bi and Giles, 2007], [Bi and Giles, 2009]. They model the financial risk associated with the returns of Box Office in the USA.

4.6 Value-at-Risk Versus Expected-Shortfall

This subsection makes a somewhat practical side-by-side comparison between ES and VaR. It does so by discussing some applications and research results where ES emerges superior. First and foremost, ES is superior to VaR as a measure of market risk. [Liang and Park, 2007] provide empirical evidence that confirms the theory by [Artzner et al., 1999] where they argue that ES supersedes VaR as a risk measure.

Secondly, [Trindade and Zhu, 2007] derive expressions of *parametric and non-parametric Estimators* (NPEs) of VaR and ES under random sampling and Laplace distributions. They show that the NPE exhibits greater asymptotic efficiency compared to the parametric estimator than is the case for VaR.

Furthermore, back-testing ES estimates and computation methods yield more favorable results than those for VaR. [Kerkhof and Melenberg, 2004] use a simulation study to show that tests for ES with generally acceptable low levels perform much better than tests for VaR in real-world financial

sample sizes.

In addition, [[Yamai and Yoshihara, 2005](#)] illustrate that the tail risk of VaR can create grave challenges in some particular cases where ES can be more efficient. They discuss market-stressing scenarios for concentrated credit portfolios and foreign exchange rates.

Last but not least, [[Mittnik and Yener, 2009](#)] show that the use of conventional correlations for modeling dependencies may lead to counter-intuitive behavior of risk measures such as VaR and ES in simulation-based assessments of rare event risk. They show that the phenomenon can be circumvented in the case of ES using appropriate simulation setups, but not for the VaR measure.

5 Simulations

This section presents an overview of simulations and some of the applications of simulations. It describes what a model is, and what it means to simulate a model. The evolution of simulation is briefly visited and the main types of simulations are discussed. The section concludes with some applications of simulation.

5.1 Models and Simulation at a Glance

It is not rare to find systems that cannot be directly investigated. Investigations may be difficult or even impossible due to several reasons including high cost, safety concerns, time considerations, or legal restrictions. In such cases, models may be used to study the systems in question. [White and Ingalls, 2015] describes a model as "an entity that is used to represent some other entity for some defined purpose". They are a representation of real-life phenomena using simplified and abstract constructions to investigate an actual system that is impractical or prohibitive. Models can be physical, and they can also be represented by a set of mathematical equations that describe the evolution of a system.

Simulation is an approach to investigating models experimentally. [Shannon, 1975] defines simulation as "the process of designing a model of a real system and conducting experiments with this model for the purpose either of understanding the behavior of the system or of evaluating various strategies (within the limits imposed by a criterion or set of criteria) for the operation of the system.". Thus, it involves experimenting with a system model to generate observations in an attempt to understand, summarize, and/or generalize the behaviors observed from the experiments. For models that are formulated

mathematically, the simulation may prove to be a significant tool for the investigation of the behavior of a model for a given change in the parameters of the model.

5.2 Evolution of Simulation

Simulation is believed to have stemmed from the **Buffon needle experiment** in 1777, and the subsequent contribution of Laplace to the experiment in 1812, both of which aimed at estimating the value of π [Goldsman et al., 2010]. After about a century, **William Sealy Gosset** published a paper formalizing the Students' t-distribution. Before publishing, he employed a crude form of manual simulation for the verification of his initial conjecture about the functional form of the probability density of the distribution.

The development of general-purpose electronic computers in 1981, as well as the application of Monte Carlo methods by John von Neumann, Nicholas Metropolis, and Stanislaw Ulam, among others, to the creation of the hydrogen bomb in the mid-1940s largely catapulted the advancement of simulation. Another major element in the advancement of the Monte Carlo method in the 1950s was the development of Linear Congruential Random-number Generators (LCGs) by [Lehmer, 1951] and the increased availability of electronic computers at the time. In the early 1960s, Keith Douglas Tocher developed the very first general-purpose simulator, the General Simulation Program (GSP), to systematically build a simulation of an industrial plant comprising several machines. Around the same time, Geoffrey Gordon introduced the General Purpose Simulation System (GPSS) to facilitate rapid simulation modeling of complex teleprocessing systems. Philip Kiviat also played a key role in the development of two Simulation Programming Languages (SPLs), SIMSCRIPT and General Activity Simulation Program (GASP), as well as

Kristen Nygaard and Ole-Johan Dah in the creation of SIMULA I, the most influential programming language in the history of computing.

1970 to the early 1980s marked a period of expansion to discrete-event simulation and a period of advancement to simulation programming languages (SPLs). In the area of random number generation, more attention was given to the development of generators that generated numbers with more desirable characteristics, and more advanced general methodologies were created to efficiently generate realizations from all of the probability distributions. The inception of the Winter Simulation Conference in 1976 also played a very important role in bringing researchers together for the enhancement of simulation.

5.3 Types of Simulations

Simulations may be categorized depending on the nature of experimentation and depending on the nature of the time steps of the states of the system being experimented with. Depending on the nature of the experiment, simulations may be grouped as *Physical simulation* and *Computer simulation*. Depending on the nature of the changes in states of the system, simulations may be categorized as *Discrete-event simulation*, *Continuous-event simulation*, and *Hybrid simulation*.

Physical simulation is a simulation where physical objects are used rather than the actual materials or actual setup to reduce the scale or expenditure of the experimentation. A special kind of physical simulation is *Interactive simulation* or *Human-in-the-loop simulation* which uses human operators in the simulation. Driving simulators, flight simulators and sailing simulators are all physical simulations.

Computer simulation on the other hand refers to the use of computer

software for experimenting with a model of a particular system. The other types of simulations mentioned above all fall under computer simulations.

[[Goldsman and Goldsman, 2015](#)] describe Discrete-event simulation as a simulation that is "characterized by changes in the simulation's state at discrete time points." That is, this type of simulation takes into consideration models of systems where the change in the values of the model state is discrete. Queuing systems, inventory systems, and manufacturing systems are examples of systems that can be modeled and simulated using discrete-event simulation.

On the other hand, Continuous simulation consists of a model whose states or variables continuously change over time, [[Bandyopadhyay and Bhattacharya, 2014](#)]. Thus, simulation is based on continuous time, making use of integration and differential equations.

The combination of Discrete-event simulation and continuous simulation is Hybrid simulation or Combined simulation. It involves a mix between the two simulation types and results in the numerical integration of differential equations between two sequential events to reduce the number of discontinuities, [[Giambiasi et al., 2001](#)].

This project emphasizes computer simulation in the computation of Value-at-Risk and Expected-Shortfall.

5.4 Application fields for Simulation

The applications of simulation are vast. It is useful in numerous sectors including manufacturing, transport and logistics, engineering and construction, environment, healthcare and medicine, networking, and so on. [[Bandyopadhyay and Bhattacharya, 2014](#)]. [[Bandyopadhyay and Bhattacharya, 2014](#)] outline the various applications of simulations in each of the sectors according to the annual Winter Simulation Conference (WSC).

A considerable amount of applications of simulations have been made in finance. Most of the applications focus on estimating or forecasting a variety of prices and variables such as option prices, derivative sensitivities, interest rates, inflation, exchange rates, stock prices, risk premiums, and numerous forms of risk. In particular, a huge effort has been put into using simulation to estimate market risk. VaR and ES, being among the most popular methods for calculating market risk naturally come off as eligible for applying simulations to their computation.

However, most simulation methods employed in their calculation are often time slow and sometimes produce inaccurate results. The next section describes some simulation methods that prove computationally efficient and provide better accuracy.

6 Efficient Simulation Approaches

Monte Carlo (MC) simulation is one of the most popular numerical computational methods employed in engineering, physical sciences, and social sciences including economics and finance. Besides VaR and ES, the method has been applied in estimating other indicators in Finance and Economics. The Monte Carlo method is flexible and easily adaptable to solving many problems, however, its main shortfall is its relatively slow rate of convergence, and inaccuracy in some cases. Ample research has been dedicated to exploring alterations of the method, and the creation of entirely new methods as an attempt to remedy the main challenges of the classic Monte Carlo method.

This section explores alternative simulation approaches to computing VaR and ES in the literature generally viewed as improvements over the Monte Carlo method. The section clusters these efficient simulation approaches into two categories: The first category as discussed in **Section 6.1** below considers research that aims at improving the convergence time relative to the classic Monte Carlo Method by employing alternative methods of simulating the underlying model, or in other words, by using alternative models to generate simulated data. The second category as discussed in **Section 6.2** below considers simulation methods that strive to improve the likeness of the simulated model to the empirical underlying distribution.

6.1 Methods Based on Alternative Simulated Data Generation

This subsection describes the random number-generating processes used in classic Monte Carlo and in more efficient methods such as the Quasi-Monte

Carlo, and the Randomized Quasi-Monte Carlo methods. The last part of the subsection covers the literature on mainstream applications of the Randomized Quasi-Monte Carlo method in computing VaR and ES. Brief descriptions of the methodology in each application, together with their findings about the efficiency of their method relative to the classic Monte Carlo method are highlighted.

6.1.1 Monte Carlo Method - Pseudorandom Number Generator (PRNG)

The method through which a sequence of random numbers is generated largely affects the efficiency of estimates computed. Classical MC uses Pseudorandom Number Generators (PRNGs) to generate the sequence of numbers used in simulation. Generally, PRNGs generate sequences of numbers by initializing on values from within a finite set of states, \mathbf{S} with a function $f : \mathbf{S} \rightarrow \mathbf{S}$ producing the next state such that

$$\mathbf{S}_n = f(\mathbf{S}_{n-1}), n = 1, 2, \dots$$

The initializing state (S_0), also known as the seed is defined by the researcher or determined using an algorithm. The choice of the seed is essential since it reproduces the same sequence when the same initialization state is used.

The main types of PRNGs are the Linear Congruential Generators (LCGs) and the Mersenne Twister, the latter being the most widespread. LCGs are the simplest type of PRNG used. They create sequences of numbers according to a recursive formula with the general form

$$x_{n+1} = ax_n + c \pmod{m},$$

where a , m , and c are positive integers, and x_0 is the seed. RANDU is a specific case of LCGs where c is zero, and a and m are defined. On the other hand, the Mersenne Twister in [Matsumoto and Nishimura, 1998] is not an LCG but is classified as a twisted generalized feedback shift register that generates a sequence of word vectors, or a row vector consisting of zeros and ones, representing the integers in binary form from 0 to $(2^w - 1)$ where w is the dimension of the row vectors. Like LCG's, the Mersenne Twister uses a recursive formula of the form:

$$x_{k+n} = x_{k+m} \oplus (x_k^u | x_{k+1}^l) \mathbf{A},$$

where $k = 0, 1, \dots, n$ and m are integers with $1 \leq m \leq n$, where n is the degree of recurrence. x_k^u is the upper $w-r$ bits of x_k , and x_{k+a}^l are the lower r bits of x_{k+1} . r represents the separation point of one word and is an integer satisfying $0 \leq r \leq w - 1$. \oplus is the bitwise addition in modulo 2, and $|$ is the concatenation operator. \mathbf{A} is a constant $w \times w$ matrix with elements 0 and 1.

6.1.2 Random Number Sequence Evaluation

Several measures to evaluate sequences of random numbers generated from different methods exist. The **Kolmogorov-Smirnov (KS) Test** can be employed to examine the equality of the underlying distribution of a sample to the reference distribution. The KS test technically finds the biggest difference between the reference cumulative and the empirical distributions. The smaller the difference, the better the sample generated from the RNG. Another measure to evaluate sequences of random numbers is the **Coverage** test. **Coverage** describes how well the points generated by a RNG repre-

sent the reachable set, as measured by the equidistribution of the generated points. A lower value of coverage implies a better representation of the reachable set. Last but not least, **Discrepancy** is a quantitative measure for the direct comparison of different types of RNGs allowing for the evaluation of a sequence's approximation of the quality of a uniform distribution. A high calculated discrepancy implies that a sequence of random numbers is relatively less uniform than a low discrepancy sequence, and a uniform sequence has a discrepancy of 0.

6.1.3 Quasi-Monte Carlo Method - Quasi Random Number Generator (QRNG)

Quasi-Monte Carlo (QMC) uses a Quasi Random Number Generator (QRNG) or the so-called **low-discrepancy or uniformly distributed mod 1 sequences** in simulating samples. Unlike PRNGs, which produce successive elements of a generated sequence with equal probability of occurrence anywhere within the sampling space, low-discrepancy sequences produce elements in a highly correlated manner such that successive elements in the sequence are placed far from their previous value to ensure good coverage of the sampling space. Thus, there is a tendency for PRNGs to produce clustering of points, whereas low-discrepancy sequences are designed to avoid such outcomes. In other words, low-discrepancy sequences are designed to ensure high uniformity in their domain, while pseudorandom sequences are constructed to resemble the true behavior of random numbers. In addition, low-discrepancy sequences have an infinite period, implying that sequences from QRNG will never begin to repeat.

Using the denotations of [Franke, 2018], a sequence is defined as quasi-random or low-discrepancy if $D_N \leq \frac{c_d(\log N)^k}{N}$, where c and k are constants

independent of N , however potentially dependent on the sequence dimension,

d. D_N is the **Discrepancy** defined as follows:

$$D_N = D(x_1, \dots, x_n; \mathcal{A}) = \sup_{A \in \mathcal{A}} \left| \frac{\#\{x_i \in A\}}{n} - \text{Vol}(A) \right|,$$

where $\#\{\cdot\}$ counts the number of points x_i in A , and $\text{Vol}(A)$ is the volume of A . Intuitively, the discrepancy estimates the difference between the proportion of points of a sequence that lie in a specified subinterval of a unit hypercube and the volume of that subinterval. The calculated discrepancy will be low for a relatively uniform sequence since the two quantities are proportionally and equally distributed within the hypercube, so consequently, a uniform sequence would have a discrepancy of 0 since for all subintervals A of \mathcal{A} the probability of points lying in A is equally likely [Franke, 2018].

Other discrepancy measures are the **star-discrepancy** (D^*), and the L_2 **discrepancy**. D^* restricts \mathcal{A} to subintervals of the form $\prod_{j=1}^d [0, u_j)$, so that

$$D_n^* = D_n^*(x_1, \dots, x_n) = \sup_{a \in [0,1]^d} \left| \frac{1}{n} \sum_{i=1}^n 1_{0 \leq x_i < a} - |[0, a]| \right|.$$

With the L_2 discrepancy, the maximum absolute deviation is replaced with the root mean square deviation of the actual point density with the uniform density so, that

$$D_{L_2}(X) = \left[\int_{[0,1]^d} \left(\frac{A(X, J_{a,b})}{n} - \text{Vol}(J_{a,b}) \right)^2 da db \right]^{\frac{1}{2}},$$

where X is the generated sequence, J is a subset of $[0, 1]^d$, $\text{Vol}(J)$ is the subset volume, $A(X, J)$ is the cardinality of J , $a = (a_1, \dots, a_d)^T$, $b = (b_1, \dots, b_d)^T$ and $J_{a,b} = [a_1, b_1) \times \dots \times [a_d, b_d)$.

Several forms of QRNGs have been practically employed. A more commonly used one is the **Halton Sequence**, which largely deals with prime numbers. The **Sobol' Sequence** is another form of low-discrepancy sequences that mainly applies binary operations for sequence construction. Another form of QRNG is the **Liao Sequence**, which uses powers of prime numbers. [Franke, 2018] details the descriptions of the different forms of QRNGs.

6.1.4 Randomized Quasi-Monte Carlo (RQMC)

[Morokoff and Cafisch, 1995] affirms that using low-discrepancy sequences in Monte Carlo estimation allows for faster convergence in estimates, and the generation of sequences may require lesser computational resources than the generation of sequences from PRNGs. However, one main downside of QMC is that meaningful confidence intervals for estimates cannot be obtained and the quality of the estimates obtained cannot be ascertained. This is because, since sequences used in QMC are deterministic and highly correlated (although they simulate true randomness, every member of the sequence depends on the preceding values), and initial states do not vary (QRN sequences are identical at each generation), the independent and identically distributed conditions necessary for the central limit theorem to apply are not met [L'Ecuyer, 2018].

Randomized Quasi-Monte Carlo utilizes randomized low-discrepancy sequences to compute the RQMC estimator. Randomization here refers to a process that provides a means to calculate confidence intervals by simulating a random process to obtain sample-based error estimates [L'Ecuyer, 2018]. Stated differently, randomization transforms the deterministic sequence from QRNGs into a random sequence while retaining the low discrepancy property.

Famous randomization methods include **Random Shifting/ Rotation, Scrambling, and Finite Random Sampling** [Ökten and Eastman, 2004].

6.1.5 Efficient Value-at-Risk and Expected-Shortfall Estimation Based on Randomized Quasi-Monte Carlo

Some research has employed RQMC methods in conjunction with other efficiency-driving methodologies to produce better VaR and ES estimates than those obtained from classic MC methods. [Franke, 2018] employed the Monte Carlo and the RQMC methods to estimate single-asset and multi-asset Value-at-Risk and Expected-Shortfall estimation by modeling asset prices and returns according to a Geometric Brownian Motion and a Bootstrapping-based approach. He used several data sources including the Dow Jones Industrial Average and the Apple Inc stock from 10 years to calculate a 22-day VaR at a 99%-confidence level. He found that the convergence of RQMC estimates of VaR and ES is faster than the MC estimates for the same sample size, and estimates of VaR and ES became more conservative as sample size increased for both the GBM and bootstrapping approaches for all scenarios examined. More so, he found that RQMC methods, particularly the usage of scrambled Sobol' sequences typically produced estimates with smaller standard errors under all scenarios evaluated, and improvement in standard errors of VaR and ES estimates were more substantial when the time horizon was reduced from 22 to 8 days.

In [He and Wang, 2017], they combine RQMC with Scrambled randomization and Importance sampling to calculate VaR and ES. For the distribution of the portfolio value change, they use a Cumulative Distribution Function that is taken as an expectation of a product of the indicator function and the likelihood ratio function. They prove the consistency of VaR and ES

RQMC estimates and establish useful error bounds for the ES estimate. They find that the ES estimate efficiency is strongly tied to the efficiency of the RQMC quadrature for specific discontinuous and kink functions, suggesting that increasing the accuracy of RQMC VaR and CVaR estimates necessitates increasing the efficiency of RQMC quadrature for the two mentioned functions.

Also, [Sak and Başoğlu, 2017] combines the randomized quasi-Monte Carlo method using the Sobol Sequence as the Random Number Generator with Importance Sampling (IS) and Stratified Importance Sampling (SIS) for simulating loss probability and conditional excess under the t -copula and normal-copula market risk models. To test the performance of their methods, they use randomly created portfolios of stocks traded in the NASDAQ Stock Market in the previous 11 years. They construct 95% error bounds for various MC simulations and RQMC simulations for VaR and ES using the normal and the t -copula with t and generalized hyperbolic marginals and observe that in general, each additional variance reduction technique applied in RQMC methods tends to reduce the error bound with the SIS RQMC method yielding the smallest error bounds for all the cases except the normal copula with small portfolio sizes and probability level of 0.05. They also show the convergence of RQMC with SIS estimates to the nearly exact values as the total sample size increases and suggest that RQMC with SIS converges faster than MC with SIS.

[Tzeng et al., 2018] use RQMC to compute VaR and ES for a stock portfolio whose returns follow a highly nonlinear Markov switching stochastic volatility model. They find that the RQMC method maintains the efficiency of the QMC method while providing the ability to produce reliable confidence intervals of the simulated risk measures. Their numerical analysis shows that

RQMC offers substantial error reduction, up to two orders of magnitude, in estimating the expected returns of the Markov switching model. However, they observe that the improvements offered by RQMC diminish as the dimension of the underlying sequences increases.

6.2 Methods Based on Underlying Distributions

Most VaR and ES computation methods make assumptions regarding the distribution of risk factors affecting portfolios and portfolio returns. As established in previous sections, one of the main challenges associated with the VaR and ES estimation is the assumptions about the distribution of profit and loss, particularly the losses at the left tail of the distribution. As shown in [Fama, 1965], returns are usually non-normal, and exhibit heavy tails and high peak values. For this reason, some researchers have proposed VaR and ES estimation solutions that consider the non-normality of portfolio return distributions to purposely mimic the true nature of the distribution of portfolio returns and market factors. The first part of this subsection covers those approaches for accurately describing underlying distributions for calculating VaR and ES using Bayesian methods, and the latter part considers the use of Copula distributions to represent underlying distributions.

6.2.1 Bayesian Methods

Some researchers use parametric approaches to estimating and forecasting Value-at-Risk (VaR) and expected shortfall (ES). The parametric approach usually involves a parametric specification of the underlying distribution with the need to estimate the model parameters. Bayesian methods, albeit complicated, are convenient and widely applied to estimate unknown parameters of a model specification. In general, Bayesian methods employ the Bayesian

inference framework to approximate a distribution of interest or some statistic of the distribution of interest given the data (observations). In particular, Suppose X is a random variable from a probability distribution and θ is the parameter of the distribution such that $f(X/\theta)$ is the probability distribution of X given θ . Then the distribution of the parameter θ given the observations (random variable X) $\pi(\theta/X)$, known as the Posterior distribution can be written using Bayes' Theorem as follows:

$$\pi(\theta/X) = \frac{f(X/\theta)\pi(\theta)}{\int f(X/\theta)\pi(\theta)d\theta},$$

where $f(X/\theta)$ is the likelihood and $\pi(\theta)$ is called the prior distribution (usually an educated guess by the researcher regarding the true nature of the parameter distribution). The product of the likelihood and the prior gives the joint distribution $\pi(\theta, X)$ of the observations X and the parameter θ , and the Posterior distribution $\pi(\theta/X)$ is proportional to the joint distribution.

Often, analytical methods are not enough to estimate the parameters due to high complexity or dimensionality, hence certain algorithms are employed for their evaluation. Markov Chain Monte Carlo (MCMC) represents a class of algorithms for sampling from probability distributions that are difficult to sample from or highly dimensional. An example of such an algorithm is the Metropolis-Hastings (MH) algorithm. The algorithm generally involves choosing a function ($f(x)$) proportional to the target distribution ($P(x)$), sampling from a proposal distribution ($g(x)$), and accepting or rejecting the proposed sample according to some criteria usually dependent on ($f(x)$). It is frequently used for sampling and estimation, and some researchers find it particularly useful in approximating return distributions to compute VaR and ES.

[[Chen et al., 2012](#)] proposed a parametric model for forecasting VaR and

ES that employs an asymmetric Laplace form with time-varying shape parameter as the error distribution in a framework that models returns as a GJR–GARCH. They choose uninformative priors for the Metropolis-Hastings algorithm. Their model is applied to forecast VaR and ES one day ahead for four market indices and two exchange rate series by applying it to Yahoo! Finance data, covering ten years. The data included daily series from four international stock market indices: the S&P 500 (US); FTSE 100 (UK); All Ordinaries Index (Australia); HANG SENG Index (Hong Kong), and two exchange rates: the AU dollar to the US dollar, and the Euro to the US dollar. Their simulation study illustrated that their proposed model outperforms, or is at least highly competitive with, several popular alternatives and is a consistently conservative risk model throughout the period studied.

[[Martín et al., 2023](#)] also estimate VaR and ES by modeling the tails of an investment loss distribution as a Generalized Pareto Distribution (GPD) with Exponential, Gamma, and Stable (Normal and Cauchy) baseline distributions. They consider non-informative prior distributions to estimate the parameters of the GPD which they call the Baseline MH method (BMH). They also propose a new method based on MH that employs an informative prior distribution for the parameters of the GPD which they call the Informative Prior Baseline MH method (IPBMH). Their simulation study showed that IPBMH provided the most accurate, less skewed, and precise estimates for any baseline distribution or chosen parameter. However, all three methods provided similar results as the sample size increased.

Another Bayesian approach often employed to estimate VaR and ES involves Importance Sampling (IS). [[Tokdar and Kass, 2010](#)] described IS as a Monte Carlo method to calculate an expectation of a target distribution by approximating it with the weighted average of random draws from another

distribution. It is often applied when the target distribution is unknown or difficult to sample. Generally, the idea behind IS is the following: Suppose $p(x)$ is the target probability distribution of X and the goal is to compute the expectation of X under $p(\cdot)$ ($E_p(X)$). Importance Sampling makes it possible to calculate ($E_p(X)$) even when it is impossible to sample from $P(x)$ with:

$$E_p(X) = \sum x p(x) * \frac{q(x)}{q(x)} = \sum x \frac{p(x)}{q(x)} * q(x) = E_q \left(X * \frac{p(x)}{q(x)} \right),$$

where $\frac{p(x)}{q(x)}$ is called the Importance weights, such that $E_p(X)$ is equal to the probability-weighted mean of values sampled from $q(x)$, the Importance density. A major consequence of using the IS method is that with the right choice of Importance density, IS produces estimates with reduced variance. Another advantage of the IS method is that weighting allows for sampling from the more relevant regions of the target distribution, particularly the tails of the loss distribution in the case of estimating VaR and ES.

[Hoogerheide and van Dijk, 2010] propose an algorithm for the estimation of the Bayesian VaR and ES estimators using a novel adaptive importance sampling method for the *Quick Evaluation of Risk using Mixture of approximations* (QERMit). Returns are modeled as an empirical GARCH model with Student's t distribution errors, and they approximate the optimal importance density to generate multi-step 'high loss' scenarios. An empirical analysis used S&P 500 daily log returns to estimate a 10-day-ahead VaR and ES at a 99% confidence level. They conclude from their results that the proposed QERMit approach more accurately estimates VaR and ES given the same amount of computing time, or, equivalently, requires less computing time to achieve the same numerical accuracy.

More so, *Extended Quick Evaluation of Risk using Mixture of t approx-*

imations (EQERMit) of [Borowska et al.,] calculates VaR and ES in non-linear, non-Gaussian state space models of the stochastic volatility model where they consider disturbances following Normal and Student's distributions. They apply Importance Sampling in which they approximate an Optimal Candidate Distribution (OCD) that allocates half the total mass of the OCD in the loss tail. The results from a simulation study showed that their EQERMit model produced estimates with higher precision compared to others, and illustrated the superiority of the IS-based methods.

Lastly, [Teng, 2023] proposes an efficient Importance Sampling scheme applicable for calculating the VaR and ES of a Quadratic Portfolio with t-distributed risk factors. Numerical experiments from the study showed that the proposed importance sampling estimator produced estimates with smaller variances than the standard estimator. The study also found that substantial computing time was saved with the proposed novel importance sampling estimator.

6.2.2 Methods Based on Copula Distributions

Multivariate Normal Distributions are appealing because the association between two or more random outcomes can be fully described knowing only the marginal distributions and the correlation coefficient [Frees and Valdez, 1998]. However, as stated earlier, the joint distribution of asset returns is usually non-normal. Hence, the correlation coefficient, intimately related to the multivariate Gaussian distribution, may not correctly represent the dependence structures between assets. An alternative to approximate the joint distribution of portfolio asset returns to capture the dependence structure between asset returns efficiently is by using Copula distributions.

Operationally, a Copula is a multivariate distribution function defined on

the unit cube $[0, 1]^n$, with uniformly distributed marginals [Embretchts et al., 2001]. Mathematically, [Embretchts et al., 2001] defines an n-copula as a function C from $[0, 1]^n$ to $[0, 1]$ with the following properties:

- For every \mathbf{u} in $[0, 1]^n$, $C(u) = 0$ if at least one coordinate of \mathbf{u} is 0, and $C(u) = U_K$ if all coordinates of \mathbf{u} equal 1 except U_K .
- For every \mathbf{a} and \mathbf{b} in $[0, 1]^n$ such that $a_i \leq b_i$ for all i , $V_C([a, b]) \geq 0$.

Where

$$V_C([a, b]) = \sum sgn(\mathbf{c})H(\mathbf{c})$$

is the H-volume of an n-box $[\mathbf{a}, \mathbf{b}]$ ($a_k \leq b_k \forall k$) whose vertices are in the domain of H (Dom H), with H a real function of n variables such that $DomH = S_1 \times \dots \times S_n$ for S_1, \dots, S_n nonempty subsets of the extended real line $[-\infty, \infty]$. $sgn(\mathbf{c})$ is given by

$$sgn(\mathbf{c}) = \begin{cases} 1 & \text{if } c_k = a_k \text{ for an even number of } k\text{'s,} \\ -1 & \text{if } c_k = a_k \text{ for an odd number of } k\text{'s.} \end{cases}$$

An important result regarding copulas that facilitates its application is **Sklar's Theorem**. It guarantees the existence of an n-copula for an n-dimensional distribution function with n-marginals, and the uniqueness of the n-copula for continuous marginals.

In the literature on VaR and ES estimation, the main families of copulas that are widely used are elliptical and Archimedean copulas. The former are copulas derived from elliptical distributions such as the Gaussian and Student's t copulas. The latter include the Frank, Clayton, and Gumbel copulas which are more suitable for financial contexts since they better accommodate asymmetric dependence structures [Sumbhoolaul, 2008].

[Zhu et al., 2016] presents a Gaussian and a Student's t copula-based GARCH framework that uses Markov-switching to model regimes of high and low dependence. They adopt a Bayesian approach with non-informative priors to estimate the model parameters using a Metropolis-Hastings algorithm. They undertake an empirical study in which they use a data set comprising the Stock Exchange of Thailand index (SET), Hang Seng Index (HSI), Brent oil spot price (OIL), rubber commodity price (Rubber), and rice commodity price (RICE), for the period July 2008 to April 2015. They find evidence suggesting that the Markov switching copula with Student- t performs better than the Gaussian copula. They also find that the multivariate Student- t copula parameters with regime switching conform with high dependence during market downturns and low dependence during market upturns.

More so, [Sumbhoolaul, 2008] explores copula-based approaches for computing VaR and ES using maximum likelihood estimation (MLE), inference function for margins (IFM), and canonical maximum likelihood (CML) to estimate the copula parameters. The Frank and the Gaussian copulas with normal and Student's t -distributed marginals were considered in a simulation study. The results show that the VaR and ES estimates obtained from the three estimation methods were generally similar, but the estimates obtained from the MLE were superior.

Also, [Dharmawan, 2013] models the joint distribution of stock prices using pairwise bivariate copula to generate different portfolio loss distributions from both Elliptical and Archimedean copula families with each return series fitted semi-parametrically using a piecewise distribution with generalized Pareto tail, and computes VaR and ES at 95% and 99% confidence levels over a one-day horizon.

Lastly, an importance-sampling technique for the VaR and ES estima-

tion for a credit portfolio modeled by a multifactor normal copula and a multifactor student's t copula is proposed in [Reitan and Aas, 2010].

The next section presents another efficient VaR and ES simulation estimation method based on accurately approximating the underlying distribution of portfolio loss. This method is based on the Gaussian Mixture Model (GMM) and will be used for the empirical study of this thesis.

7 Gaussian Mixture Model (GMM) as an Alternative to Geometric Brownian Motion (GBM)

Geometric Brownian Motion is one of the most commonly used processes in simulating stock prices. Its main weaknesses lie in the assumptions that stock prices are log-normally distributed and that the future behavior of a portfolio is fully reliant on past information about the underlying financial instruments. In many cases, stock returns have been observed to be fat-tailed and skewed. Mathematically, GBM stipulates that stock price returns follow a normal distribution. That is, $r_t = \ln \frac{S_t}{S_{t-1}} \sim \Phi(\mu, \sigma)$, where R_t and S_t are the return and price processes respectively. This stipulation implies that

$$\ln \left(\frac{S_t}{S_{t-1}} \right) = \mu \delta t + \sigma \epsilon \sqrt{\delta t},$$

where δt is the period and $\epsilon \sim \Phi(0, 1)$, such that the first term signifies the drift and the second represents the diffusion effect. Rearranging the above allows to simulate stock price paths as follows:

$$S_t = S_{t-1} \exp(\mu \delta t + \sigma \epsilon \sqrt{\delta t}).$$

Supposing a portfolio of N_s stocks, Monte Carlo simulation on GBM using PRNGs or QRNSs can be used to generate future stock prices by expanding the above equation in the following form:

$$\begin{bmatrix} S_{1_t} \\ \vdots \\ S_{N_{S_t}} \end{bmatrix} = \begin{bmatrix} S_{1_{t-1}}(1 + \mu_1 \delta t) \\ \vdots \\ S_{N_{S_{t-1}}}(1 + \mu_{N_S} \delta t) \end{bmatrix} + \begin{bmatrix} S_{1_{t-1}} \sigma_1 \epsilon_1 \sqrt{\delta t} \\ \vdots \\ S_{N_{S_{t-1}}} \sigma_{N_S} \epsilon_{N_S} \sqrt{\delta t} \end{bmatrix},$$

$\eta = \{\epsilon_i\}_1^{N_S}$ is a matrix of random variables from a Standard Normal distribution with N_S columns and T rows.

The i_{th} simulated portfolio value becomes

$$S_{p_t} = [\omega_1, \dots, \omega_{N_S}] \times \begin{bmatrix} S_{1_t} \\ \vdots \\ S_{N_{S_t}} \end{bmatrix},$$

where ω_i are the weights.

The return from the i^{th} simulation is given by $r_p^{i*} = \ln\left(\frac{S_{p,T}^{i*}}{S_{p,0}}\right)$, and the α -percentile of these simulated returns is $Var_{T,1-\alpha}$.

Due to the assumptions surrounding the GBM model, risk estimation employing GBM may not produce empirically oriented estimates. Hence, [Seyfi et al., 2021] proposes an efficient Monte Carlo method for simulating non-normal stock prices from a GMM. Back-testing procedures show that the model is accurate and performs better than other popular VaR and ES calculation methods. They also illustrate that the GMM model is more efficient than other Monte Carlo applications for measuring risk.

The following section describes GMMs and presents the setup for a GMM.

7.1 Gaussian Mixture Model (GMM)

[Seyfi et al., 2021] describes a Gaussian Mixture Model as "a generative algorithm used for predicting the stock market and generating new data.". Generative in the sense that it can adaptively learn the nature of the data and create unseen samples. It can also handle complicated and intractable data by learning the data structures. An important feature of GMMs is that they divide data into clusters and independently simulate data points from each cluster. GMMs are advantageous for modeling returns since they can model future outcomes to any likely scenario and make no normality assumptions about the distribution of returns for two or more components [Bishop, 2006]. Studies by [Kon, 1984] also show that the properties of GMMs make them suitable for approximating the distribution of data that exhibit heavy tails and extreme values. Another advantage of GMMs is the power to fit different distribution shapes by altering component weights and other distribution parameters such as mean and standard deviation.

7.1.1 GMM Setup

GMMs are a combination of the densities of Gaussian distributions. In other words, the density of a GMM is the weighted sum of a finite number of Normal densities. Mathematically, a GMM is the following:

$$P(x) = \sum_{i=1}^{N_c} \omega_i \Phi(x|\mu, \Sigma_i),$$

such that N_c is the number of components, ω_i 's are the weights which add up to one, μ_i 's are the mean vectors and Σ_i 's are the covariance matrices for

$i = 1, 2, \dots, N_c$. $\Phi(x|\mu, \Sigma_i)$ is the Normal density function as follows:

$$\Phi(x|\mu, \Sigma_i) = (2\pi)^{-\frac{k}{2}} \det(\Sigma_i)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(x - \mu_i)' \Sigma_i^{-1} (x - \mu_i)\right\},$$

with $x \in R$ being the input data.

The next step involves the estimation of the parameters of the model as follows.

7.1.2 Estimation of Parameters

The parameter of the model is $\theta = \omega, \mu, \Sigma$. Let $p(z_j = 1|x_i)$ denote the probability of x_i belonging to the j th Gaussian distribution, where the sequence $z = z_1, \dots, z_{N_c}$ are the latent variables from a binary distribution, $z_i \in 0, 1$, with $\omega_j = p(z_j = 1)$. Recall that GMM assumes that each data sample belongs to a single cluster, thus, only a single z_i can take the value one. The latent variable probability measure is given as follows:

$$p(z) = \prod_{j=1}^{N_c} p(z_j = 1)^{z_j} = \prod_{j=1}^{N_c} \omega_j^{z_j},$$

such that the conditional probabilities of samples conditional on the latent variables are:

$$p(x_i|z) = \prod_{j=1}^{N_c} p(x_i|z_j = 1)^{z_j} = \prod_{j=1}^{N_c} \Phi(x|\mu_j, \Sigma_j)^{z_j}.$$

Given that $p(x_i, z) = p(x_i|z)p(z)$, then the marginal distribution of $p(x_i, z)$ is given by:

$$p(x_i) = \sum_z p(x_i|z)p(z) = \sum_{j=1}^{N_c} p(x_i|z_j) p(z) = \sum_{j=1}^{N_c} \omega_j \Phi(x_i|\mu_j, \Sigma_j).$$

Using Bayes rule, the *responsibility* function is computed as:

$$p(z_j = 1|x_i) = \frac{\omega_j \Phi(x_i|\mu_j, \Sigma_j)}{\sum_{i=1}^{N_c} \omega_i \Phi(x_i|\mu_i, \Sigma_i)} = r_{ij}.$$

To calibrate the parameters, the following algorithm is used:

1. Initializing Step: Randomly assign $\theta = \omega, \mu, \Sigma$.
2. Expectation Step: calculate the *responsibility* functions by taking an expectation form z_j . That is; $p(z_j|x_i, \theta) = r_{ij}$.
3. Maximization Step: the parameters are updated using the following:

$$\omega_j = \frac{\sum_{i=1}^N r_{ij}}{N}, \mu_j^{new} = \frac{\sum_{i=1}^N r_{ij} x_i}{\sum_{i=1}^N r_{ij}},$$

$$\Sigma_j^{new} = \frac{\sum_{i=1}^N r_{ij} (x_i - \mu_j)(x_i - \mu_j)'}{\sum_{i=1}^N r_{ij}},$$

where N is the total number of data used as input.

4. Repeat 2 and 3 until convergence.

7.1.3 Generation of new samples

As mentioned, a generative model learns the data structure and uses the "experience" to generate new data. In generating new data, the GMM samples from each cluster by $\Phi(x|\mu, \Sigma_i)$, and the proportion of the number of samples from each distribution is determined by its weight ω_i . This ensures that the data generated from the joint distribution resembles the historical data.

Suppose $p(x)$ is any pdf, and let N_T denote the total generated samples. Let $N_0 \leq N_T$ denote the number of samples less than an arbitrary value x_0 .

Then,

$$\lim_{N_T \rightarrow \infty} \frac{N_0}{N_T} = \int_{-\infty}^{x_0} p(x) dx.$$

For GMM, let N_0^* be the number of GMM samples less than x_0 . Then $N_0^* = \sum_{i=1}^{N_c} N_0^i$, where N_0^i is the number of samples less than x_0 in each Normal distribution. The equation

$$\lim_{N_T \rightarrow \infty} \frac{N_0^*}{N_T} = \int_{-\infty}^{x_0} \sum_{i=1}^{N_c} \omega_i \Phi(x|\mu, \Sigma_i) dx$$

shows that new samples conform to the distribution of the Gaussian mixture.

Hence, the following algorithm can be used to simulate a GMM.

- The number of samples from each cluster can be chosen as $\omega_i N_T$ for $i = 1, \dots, N_c$.
- Generate $\omega_i N_T$ samples for the i_{th} cluster.
- Integrate samples from all clusters.

7.2 Calculating Value-at-Risk and Expected-Shortfall Based on GMM

As shown by [Seyfi et al., 2021], GMM can generate new samples from the joint mixture of Gaussian distributions. To generate a path of say \mathbf{T} new samples, $\omega \mathbf{T}$ samples are allocated to cluster i , and the new samples say $\{r_1^*, \dots, r_T^*\}$ is the set of possible returns for the stock. Stock prices, say $\{S_1^*, \dots, S_T^*\}$ are calculated by $S_t^* = S_{t-1}^* \exp r_t^*$ for $t = 1, 2, \dots, T$. m stock price paths are then generated by repeating the algorithm m times.

Recall that VaR measures the risk of loss for a given significance level (α) and time horizon t . That is, VaR at α for t solves $\Pr(S \geq VaR_{\alpha,t}) =$

$1 - \alpha$. Since VaR measures the loss in a particular time horizon, any form of Monte Carlo Simulation prioritizes the last returns simulated. Denote the last returns as $P_T^{*i}, i = 1, \dots, m$, the VaR is then the α -percentile of all the generated returns P_T^{*i} .

The returns matrix for m paths and T evaluation days is given by :

$$\begin{bmatrix} r_1^{*1} & \dots & r_T^{*1} \\ \vdots & \ddots & \vdots \\ r_1^{*m} & \dots & r_T^{*m} \end{bmatrix}.$$

The path for the stock prices are then generated using $S_t^* = S_{t-1}^* \exp r_t^*$ as:

$$\begin{bmatrix} S_1^{*1} & \dots & S_T^{*1} \\ \vdots & \ddots & \vdots \\ S_1^{*m} & \dots & S_T^{*m} \end{bmatrix}.$$

The total return of each path is computed as $R_{GMM}^i = \ln\left(\frac{S_T^*}{S_0}\right)$.

The α -percentile of R_{GMM}^i is $VaR_{1-\alpha}$, and the ES is calculated as :

$$ES_{1-\alpha} = \frac{1}{n} \sum_{i=1}^m R_{GMM}^i \times \mathbf{I}_{\{R_{GMM}^i \leq VaR_{1-\alpha}\}},$$

where \mathbf{I} is the indicator function:

$$\mathbf{I}_A = \begin{cases} 1, & A \\ 0, & otherwise \end{cases},$$

and n is the number on ones in $\mathbf{I}_{\{R_{GMM}^i \leq VaR_{1-\alpha}\}}$.

GMM guarantees that this new matrix of returns produces the same distribution as the historical returns. A major advantage of using an approach based on GMM to calculate VaR and ES is that during crisis periods, the al-

gorithm quickly learns that clusters of correlated losses happen concurrently. This is achieved by allocating the appropriate weights to separate return clusters during turbulent periods. For instance, higher weights would be given to extreme market scenarios during a crisis period and vice versa. This would then translate to a corresponding rise or decrease in the risk measure.

VaR heavily depends on return distributions and GMM has the power to capture return distributions relatively well. However, VaR also depends on volatilities. Hence, [Seyfi et al., 2021] propose a method to adjust VaR and ES for the volatility levels and volatility periods. The proposed adjusted values are computed as:

$$VAR_{1-\alpha}^{adj} = VaR_{a-\alpha} \times \frac{\sigma_{short}}{\sigma_{long}}, ES_{1-\alpha}^{adj} = ES_{a-\alpha} \times \frac{\sigma_{short}}{\sigma_{long}}.$$

Equivalently, the stimulated returns may be adjusted by multiplying the return matrix by $(\frac{\sigma_{short}}{\sigma_{long}})$, and calculating the corresponding stock price matrix. The VaR and ES can then be calculated from the adjusted matrices.

The elements in the adjusted return matrix will be:

$$r_{sj}^{i*} = r_{sj}^{i*} \times \left(\frac{\sigma_{short}}{\sigma_{long}}\right)_{sj}.$$

For a T-horizon, a stock s_j has a return of $\sum_{t=1}^T r_{sj,t}^{i*}$. The return on the portfolio given the fixed weights $[\omega_1, \dots, \omega_{N_s}]$ is calculated as:

$$\begin{bmatrix} r_p^{1*} \\ \vdots \\ r_p^{m*} \end{bmatrix} = \begin{bmatrix} \sum_{t=1}^T r_{s1,t}^{1*} & \cdots & \sum_{t=1}^T r_{sN_s,t}^{1*} \\ \vdots & \ddots & \vdots \\ \sum_{t=1}^T r_{s1,t}^{m*} & \cdots & \sum_{t=1}^T r_{sN_s,t}^{m*} \end{bmatrix} \times [\omega_1, \dots, \omega_{N_s}]',$$

such that the i_{th} simulated value of the portfolio at T is given by:

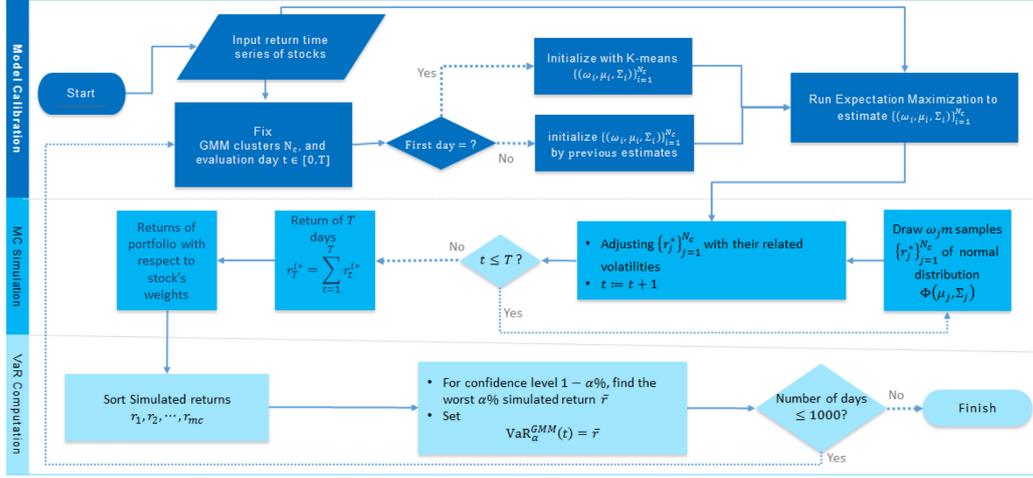


Figure 1: VaR-calculation method based on GMM. Source: [Seyfi et al., 2021]

$$S_{p,T}^{i*} = S_{p,0} \times \exp r_p^{i*}, \text{ for } i = 1, 2, \dots, m.$$

The $VaR_{T,1-\alpha}$ can then be calculated as the α -percentile of the generated price series, and the $ES_{T,1-\alpha}$ can be estimated as

$$\frac{1}{n} \sum_{i=1}^m R_{GMM}^i \times \mathbf{I}_{\{R_{GMM}^i \leq VaR_{1-\alpha}\}}, \text{ where } \mathbf{I} \text{ is the indicator function.}$$

Figure 1 illustrates the procedure for computing VaR from the model calibration to the simulation process and the calculation itself.

8 Empirical Analysis

This section presents the numerical results based on a Matlab implementation for calculating VaR and ES for a portfolio of stocks using stock return simulation based on GMM, Geometric Brownian Motion-based Monte Carlo (GBM), and Geometric Brownian Motion-based Randomized Quasi-Monte Carlo (RQMC-GBM) with scrambled randomization. The Root-Mean-Square-Error (RMSE) goodness-of-fit test is used to ascertain how much the models are being fitted to the data. The Christoffersen Back-testing procedure is employed to investigate the validity of VaR estimates for the three models.

8.1 Data and Hyperparameters

The analysis uses Bloomberg’s historical adjusted closing price data on 15 components of the FTSE MIB stock index. The stocks include; Saipem S.p.A. (SPM IM), Eni S.p.A. (ENI IM), Intesa Sanpaolo S.p.A. (ISP IM), UniCredit S.p.A. (UCG IM), Assicurazioni Generali S.p.A. (G IM), Tenaris S.A. (TEN IM), Terna S.p.A. (TRN IM), Telecom Italia S.p.A. (TIT IM), Leonardo S.p.A. (LDO IM), BPER Banca S.p.A (BPE IM), Snam S.p.A (SRG IM), Davide Campari-Milano N.V (CPR IM), Unipol Gruppo S.p.A (UNI IM), Azimut Holding S.p.A (AZM IM), and Amplifon S.p.A (AMP IM). The logarithmic returns computed from the stock price data from 01/01/2019 to 03/31/2021 are used. **Table 3** shows the descriptive statistics of the return data. The time horizon for the models is 1 day, and VaR and ES are computed for 1000 days (steps). Specifically, the 99% daily VaR and ES of an equally weighted portfolio of stocks are calculated. The number of simulations in the models is 1000. [Seyfi et al., 2021] find that the market can be explained well

enough with 3 different Gaussian distributions, so the number of components used for the GMM is 3. **Figure 2** illustrates the historical return data of three stocks (ENI IM, ISP IM, and UCG IM) in three scatter plots, each plotted against the FTSE MIB stock index on the x-axis (up). It also shows the clustering of the same return data according to a GMM with 3 components (down). **Figure 3** also illustrates the clustering of return data of other stocks using the GMM with three components.

Figure 4 plots the frequency distribution of the stock returns of eight stocks, and the stock index. It also fits a three-component GMM to each stock and overlaps the graph with a normal distribution for comparison. It clearly demonstrates the ability of the three-component GMM to effectively capture the distribution of each stock.

The number of days considered for computing σ_{long} and σ_{short} are 252 and 60 days, respectively.

	Mean	Mode	Median	Quantiles	Variance	Skewness	Kurtosis	Maximum	Minimum
SPM IM	-5.7245e-04	0	0	-0.0792	7.9049e-04	-1.0654	13.6168	0.1161	-0.2420
ENI IM	-2.4560e-04	0	0	-0.0647	5.5328e-04	-2.1864	31.0354	0.1392	-0.2338
ISP IM	4.6073e-04	0	0	-0.0563	4.4066e-04	-1.6269	19.9492	0.0926	-0.1958
UCG IM	-1.2266e-04	0	-4.2758e-04	-0.0747	7.54856e-04	-0.6319	9.5535	0.1286	-0.1895
G IM	4.2784e-04	0	7.5656e-04	-0.0430	2.4280e-04	-1.3577	20.9417	0.1049	-0.1387
TEN IM	1.0193e-04	0	0	-0.0659	7.4481e-04	-1.1347	16.4262	0.1287	-0.2407
TRN IM	5.8407e-04	0	3.3685e-04	-0.0497	2.7774e-04	-1.7043	20.1226	0.0752	-0.1619
TIT IM	-3.4491e-05	0	0	-0.0801	6.3609e-04	-1.1230	12.9157	0.1068	-0.2040
LDO IM	-1.1975e-04	0	0	-0.0696	8.2513e-04	-0.6804	16.2749	0.1507	-0.2504
BPE IM	-2.7320e-04	0	0	-0.0759	8.8084e-04	0.2231	12.2297	0.2022	-0.1954
SRG IM	5.7531e-04	0	8.8967e-04	-0.0529	3.4670e-04	-2.8368	35.3815	0.0953	-0.2129
CPR IM	4.6256e-04	0	0	-0.0527	3.4416e-04	-0.8596	20.7857	0.1013	-0.1757
UNI IM	5.8475e-04	0	8.5127e-04	-0.0674	5.4127e-04	-0.6398	16.0230	0.1632	-0.1924
AZM IM	0.0015	0	0.0012	-0.0636	5.8243e-04	-0.6486	12.1808	0.1430	-0.1731
AMP IM	0.0014	0	0.0024	-0.0683	5.8229e-04	-1.2303	15.7065	0.1057	-0.2163

Table 3: Descriptive Statistics of the Historical Returns from 01/01/2019 to 03/31/2021

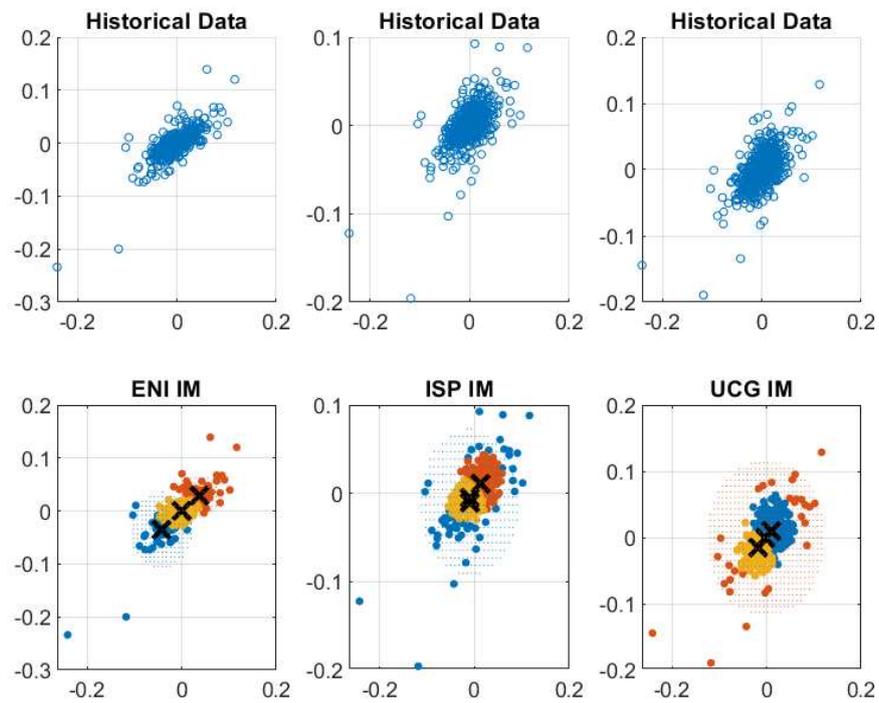


Figure 2: Scatter plots of historical returns of individual stocks(y-axis) plotted against the stock index (y-axis) (up), and their respective clustering with three components (down).

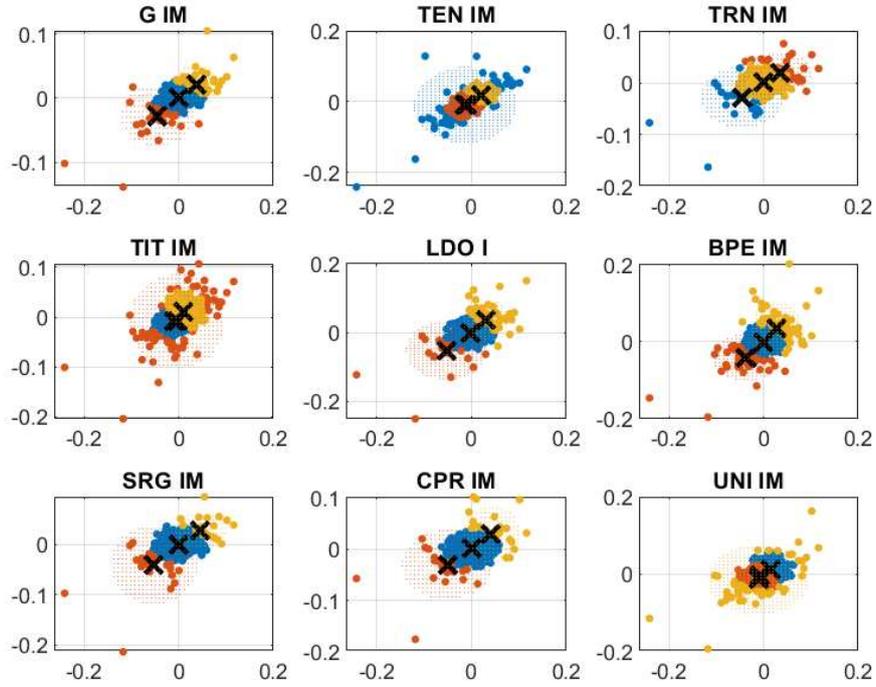


Figure 3: Clustering of stock returns against FTSE MIB stock index with three GMM components.

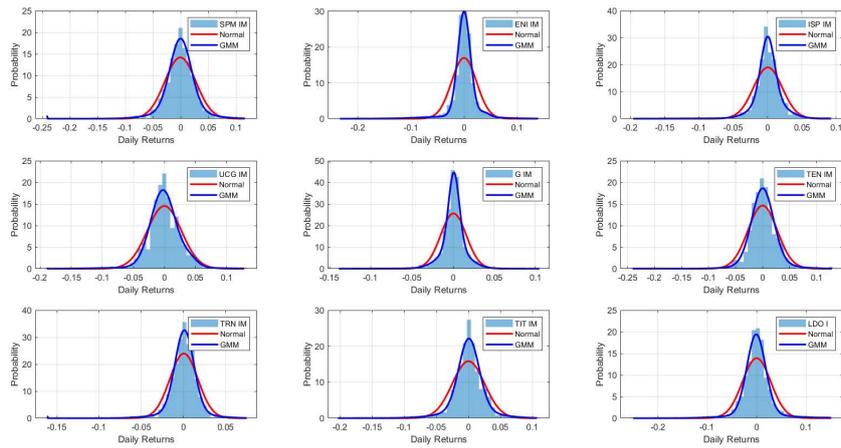


Figure 4: Frequency distribution, with Normal and 3-component GMM fittings.

8.2 Goodness-of-fit Test Results

The root-mean-square error (RMSE) goodness-of-fit test is employed to ascertain the ability of each model to fit stock return data. The RMSE test is performed for only the first time step. The average simulated return for each stock is employed in the RMSE computation. **Table 4** provides the values of each stock from the RMSE test. The RMSE values of the GMM are mostly lower than those of the GBM and RQMC-GBM, suggesting that the GMM better fits the data, a result which is supported graphically by **Figure 4**.

8.3 Portfolio VaR and ES

The VaR and ES of an equally weighted portfolio of 15 FTSE MIB stocks are computed via the three models. For each of the 3 models, the Daily and 5-day VaR and ES are computed for 1000 days at 99% and 95% confidence. **Figure 5** and **Figure 6** show the estimated 99% daily portfolio VaR and ES respectively, signifying that 1% of the portfolio returns are expected to exceed the computed daily VaR threshold, while the ES gives a more conservative estimate of potential losses that fall below the estimated VaR level. Using similar interpretation, **Figures 7** and **8** show the calculated 99% 5-day portfolio VaR and ES. In addition, the 5-day portfolio VaR and ES at 95% are shown in **Figures 9** and **10**. As seen in the illustrations, the VaR and ES estimates from GBM and RQMC-GBM almost coincide because the difference is insignificant, making it apparent that the randomization method used does not effect much change in capturing the tail distribution. However, the VaR and ES estimates based on the GMM are significantly higher than those of the other two models. This is largely due to the ability of the GMM to sample rare event returns and to reflect excess kurtosis relative to the

samples based on the GBM. Thus, using the GBM and RQMC-GBM models tends to underestimate the value of the risk measures.

Furthermore, it is evident from the graphs that ES consistently remains above VaR for any level of confidence and the number of days of evaluation. This is an indication that ES better captures tail risk compared to VaR. More so, periods of increased risk and high volatility are likely depicted by higher values of VaR and ES in those periods.

	GMM	GBM	RQMC-GBM
SPM IM	0.0281	0.0281	0.0281
ENI IM	0.0236	0.0235	0.0235
ISP IM	0.0210	0.0210	0.0210
UCG IM	0.0275	0.0275	0.0275
G IM	0.0156	0.0156	0.0156
TEN IM	0.0273	0.0273	0.0273
TRN IM	0.0167	0.0167	0.0167
TIT IM	0.0252	0.0252	0.0252
LDO IM	0.0287	0.0287	0.0287
BPE IM	0.0297	0.0297	0.0297
SRG IM	0.0186	0.0186	0.0186
CPR IM	0.0186	0.0185	0.0185
UNI IM	0.0233	0.0233	0.0233
AZM IM	0.0241	0.0242	0.0242
AMP IM	0.0240	0.0242	0.0242

Table 4: RMSE Statistics

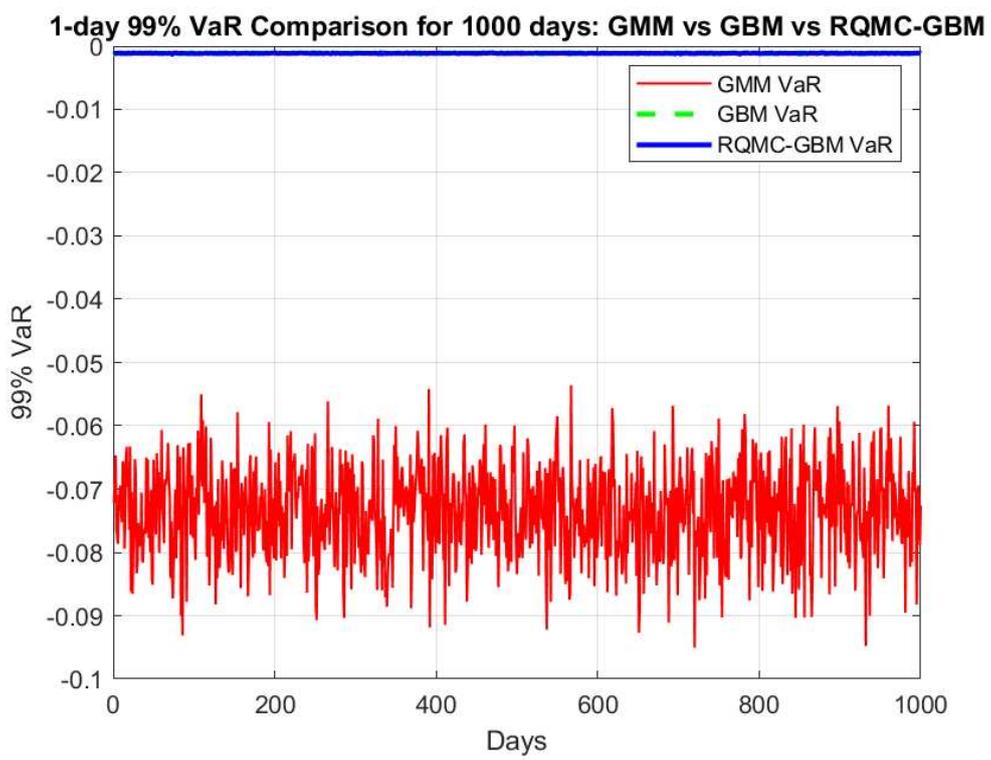


Figure 5: Daily 99% Portfolio Value-at-Risk (VaR) Comparison for 1000 days: GMM vs GBM vs RQMC-GBM

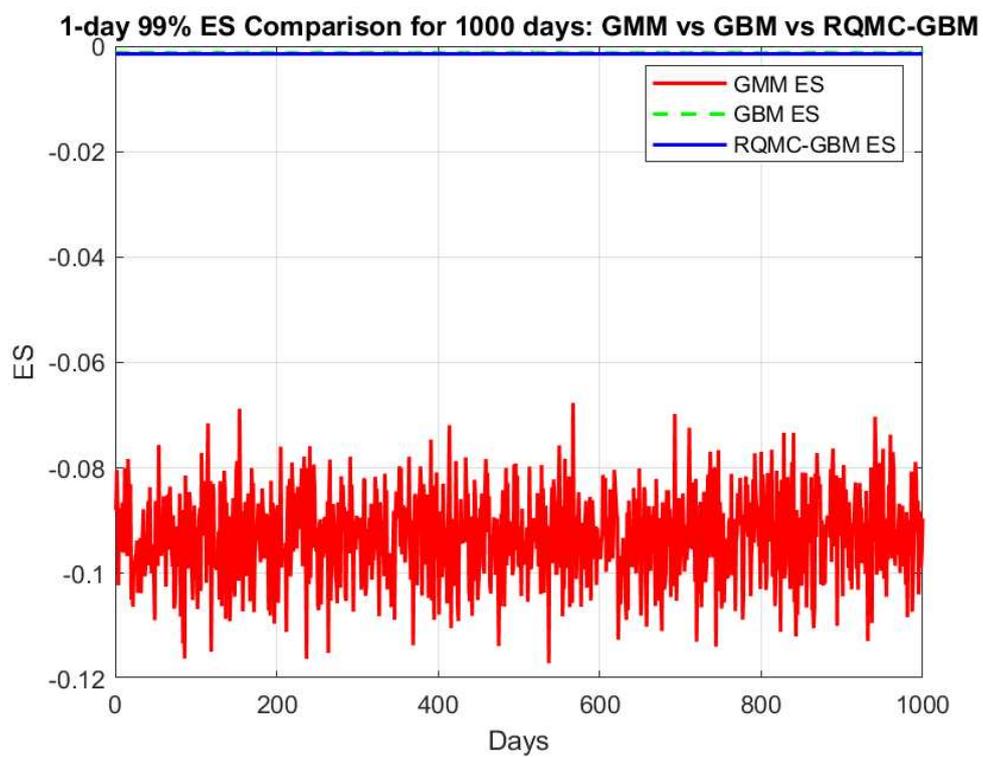


Figure 6: Daily 99% Portfolio Expected-Shortfall (ES) Comparison for 1000 days: GMM vs GBM vs RQMC-GBM

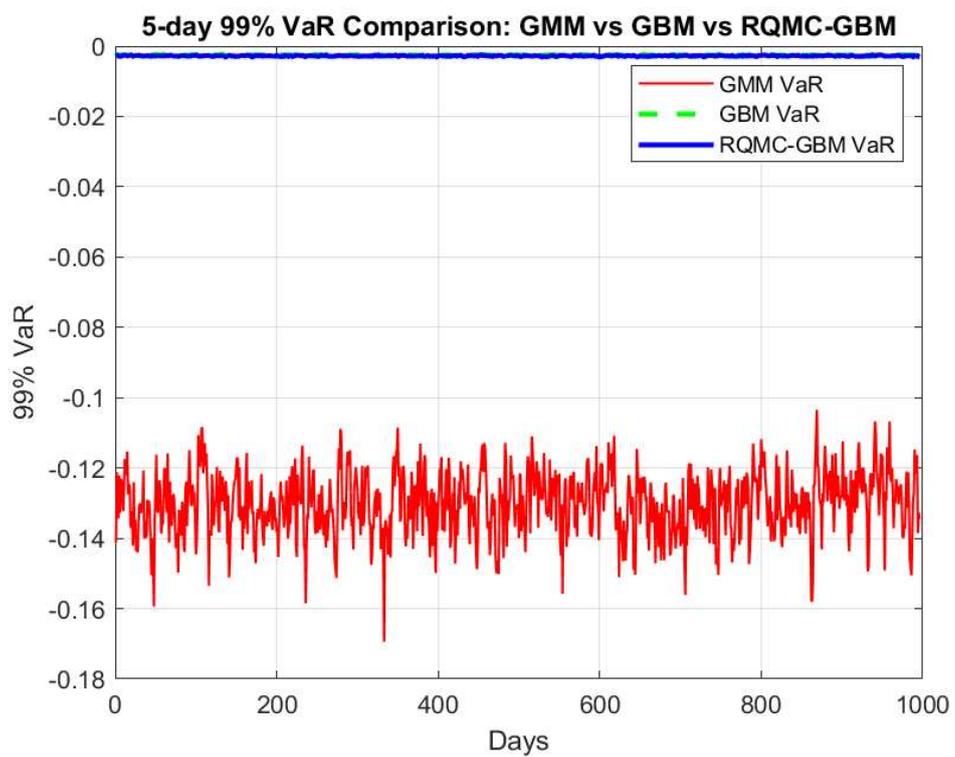


Figure 7: 5-day 99% Portfolio Value-at-Risk (VaR) Comparison: GMM vs GBM vs RQMC-GBM

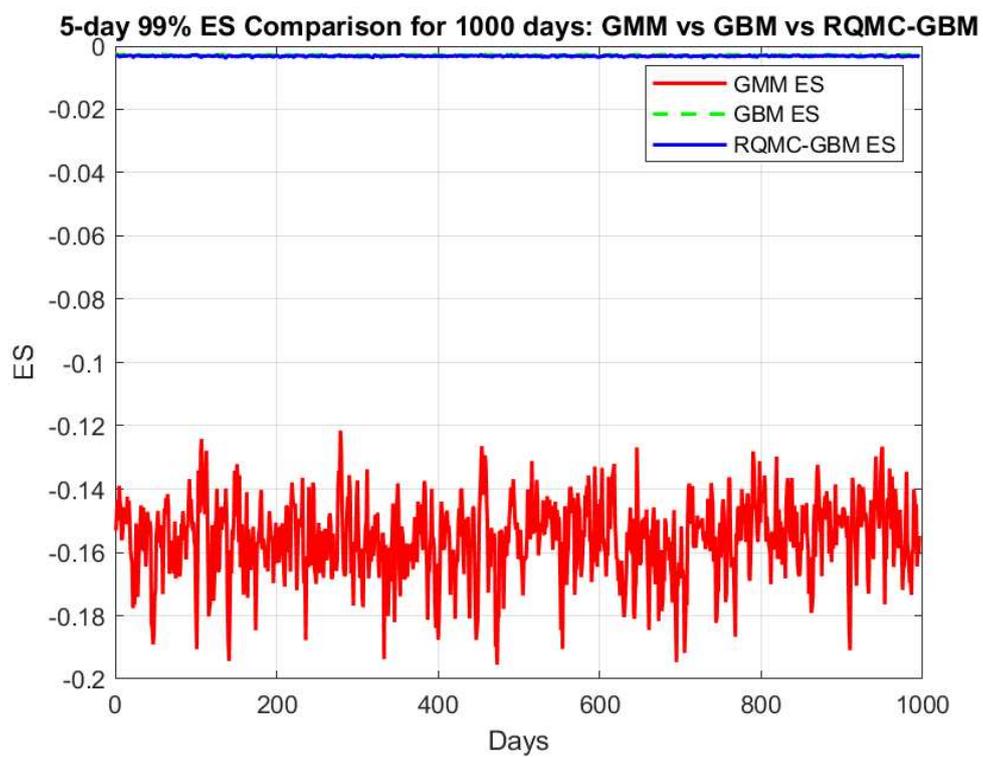


Figure 8: 5-day 99% Portfolio Expected-Shortfall (ES) Comparison: GMM vs GBM vs RQMC-GBM

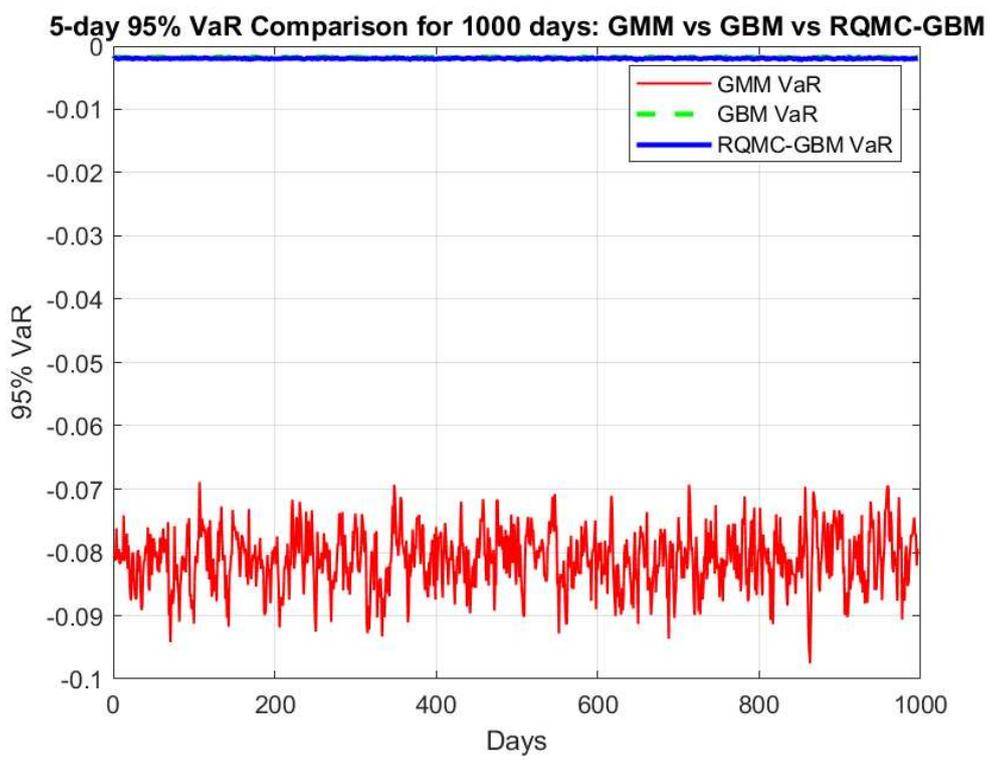


Figure 9: 5-day 95% Portfolio Value-at-Risk (VaR) Comparison: GMM vs GBM vs RQMC-GBM

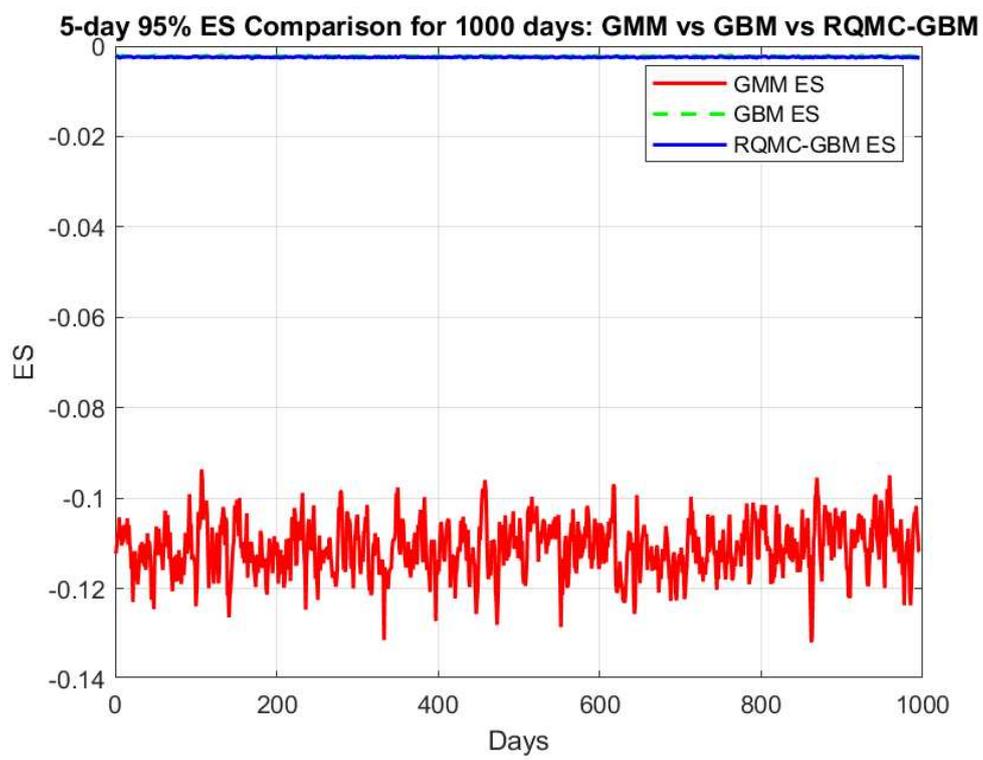


Figure 10: 5-day 95% Portfolio Expected-Shortfall (ES) Comparison: GMM vs GBM vs RQMC-GBM

	Unconditional Coverage	p-value-uc	Independence	p-value-ind
GMM	1.7169	0.1901	0.0309	0.8604
GBM	∞	0	0.2325	0.6297
RQMC-GBM	∞	0	0.0587	0.8085

Table 5: Christoffersen Test Results

8.4 Christoffersen VaR Validity Test Results

The validity of the VaR and the ES estimates is ascertained using the Christoffersen back-testing procedure proposed by [Christoffersen, 1998]. According to this procedure, the VaR estimates must satisfy both unconditional coverage (UC) and independence properties (IND) of exceeding returns to pass the test. The VaR results are valid if the p-value for both the unconditional coverage and the independence statistics is greater than 0.01. According to the test, the VaR from employing the 3-component GMM, and the ES by extension turn out to be the only valid estimates. **Table 5** provides the test statistic of each of the three models and their p-values.

9 Conclusion

The traditional methods of computing market risk measures, namely VaR and ES, are reviewed. Simulation methods based on efficient data generation methods such as randomized Quasi-Monte Carlo methods, and methods based on accurate dependence structure modeling of underlying distributions such as Gaussian Mixture Models are discussed. A simulation study to analyze the performance of VaR and ES using Gaussian mixture models, Monte Carlo based on geometric Brownian motion, and Randomized Quasi-Monte Carlo based on geometric Brownian motion is conducted.

A goodness-of-fit test suggests that the GMM better fits historical data, and a VaR backtesting method affirms the validity of the GMM VaR and ES estimates. In addition, GMM can replicate the dependence structure of stock returns by preserving the non-linear correlation between individual stock returns.

It is worthwhile noting that [Seyfi et al., 2021] compares the computing time between GMM and other VaR estimation methods using Python and finds that executing GMM is faster than GBM by a large margin. However, my simulation study implements the calculations in Matlab (Appendices A, B, and C correspond to the code for each model, while D and E are the codes for the Figures and Tables respectively). Although the execution time for GMM is the slowest, the difference between execution times is nearly insignificant.

An interesting line of research for future works might be to use portfolio optimization methods such as Modern Portfolio Theory to estimate optimal portfolio weights and then employ GMM.

Appendix A

```
% GMM CODE
clc;close all;clear all;
%% Historical Data

Data = xlsread('Stock_prices_adjusted.xlsx','Normal_Abnormal_CashChanges');
Data(:,1) = [];
Data = Data(3523:4109,:); % Data from 01/01/2019 to 31/03/2021
Num_observations = size(Data,1); % Number of observations
Num_stocks = size(Data,2); % Number of Stocks
Stock_names(1:Num_stocks) = ["SPM IM" "ENI IM" "ISP IM" "UCG IM" "G IM"...
"TEN IM" "TRN IM" "TIT IM" "LDO I" "BPE IM" "SRG IM" "CPR IM" "UNI IM"...
"AZM IM" "AMP IM"];
Daily_returns = diff(log(Data)); % Calculating daily log returns
Stock_weights = ones(Num_stocks, 1)/Num_stocks;
% Equal portfolio weight

%% Descriptive Statistics of return series

return_mean = mean(Daily_returns);
% mean of the returns in each column
return_var = var(Daily_returns) ;
% daily variance of the returns
return_correlation = corr(Daily_returns);

Sigma_long = sqrt(var(Daily_returns(1:252,:)));
% Standard deviation of long period(1 working year)
Sigma_short = sqrt(var(Daily_returns(end-59:end,:)));
% Short period standard deviation (60 days)
Adj = Sigma_short./Sigma_long;

%% Initializing with K-means, fitting a Gaussian Mixture Model (GMM),
% and Calculating VaR and ES for first evaluation day

% Parameters
Num_components = 3; % Number of GMM components
```

```

Num_steps = 1000; % Number of times to run the Expectation Maximization
% algorithm.

% Initializing matrices to store GMM parameters
Component_means = zeros(Num_components, Num_stocks, Num_steps);
Component_cov = zeros(Num_stocks, Num_stocks, Num_components, Num_steps);
Component_weights = zeros(1, Num_components, Num_steps);

% Fitting the GMM using K-means initialization
Initial_means = kmeans(Daily_returns, Num_components);
% k-means clustering
Gmm = fitgmdist(Daily_returns, Num_components, 'Options',...
statset('MaxIter', 1000),'RegularizationValue', 1e-6, 'Start', Initial_means);

% Storing the first parameters from EM algorithm
Component_means(:,:,1) = Gmm.mu;
Component_cov(:,:,:,1) = Gmm.Sigma;
Component_weights(:,:,1) = Gmm.ComponentProportion;

% Simulating 1000 samples on the first day of calibrating of the model
Num_sim = 1000;
Simulated_returns_Gmm1 = zeros(Num_sim, Num_stocks, Num_steps);
for i = 1:Num_sim
    % Generating Samples from the GMM
    Chosen_component_idx = find(mnrnd(1, Component_weights(:,:,1)));
    % Choose component based on weights
    Chosen_mean = Component_means(Chosen_component_idx,:,1);
    Chosen_cov = Component_cov(:,:,Chosen_component_idx,1);
    Sim_returns = mvnrnd(Chosen_mean,Chosen_cov); % Generate return
    Simulated_returns_Gmm1(i,:,1) = Sim_returns.*Adj; % Adgusted stock
    % returns
end

% Calculating VaR and ES for first evaluation day
Simulated_port_ret_Gmm1 = zeros(Num_sim,Num_steps);
Simulated_port_ret_Gmm1(:,1) = Simulated_returns_Gmm1(:,:,1)*Stock_weights;
Sorted_port_ret_Gmm1 = zeros(Num_sim,Num_steps);

```

```

alpha1 = 0.01;
alpha2 = 0.05;

Sorted_port_ret_Gmm1(:,1) = sort(Simulated_port_ret_Gmm1(:,1));

VaR_GMM_alpha1day1 = zeros(1,Num_steps);
ES_GMM_alpha1day1 = zeros(1,Num_steps);

VaR_GMM_alpha2day1 = zeros(1,Num_steps);
ES_GMM_alpha2day1 = zeros(1,Num_steps);

VaR_GMM_alpha1day1(1) = quantile(Sorted_port_ret_Gmm1(:,1), alpha1);
ES_GMM_alpha1day1(1) = mean(Sorted_port_ret_Gmm1(Sorted_port_ret_Gmm1(:,1)...
    <= VaR_GMM_alpha1day1(1),1));

VaR_GMM_alpha2day1(1) = quantile(Sorted_port_ret_Gmm1(:,1), alpha2);
ES_GMM_alpha2day1(1) = mean(Sorted_port_ret_Gmm1(Sorted_port_ret_Gmm1(:,1)...
    <= VaR_GMM_alpha2day1(1),1));
%% Model Calibration, fitting GMMs, and Calculating VaR and ES for
% subsequent evaluation days

% Using the parameters of previous calibrations to run EM for the next
% iterations
for t = 2:Num_steps
    % Fitting the GMM using the previous parameters
    Gmm = fitgmdist(Daily_returns, Num_components, ...
        'Start', struct('mu', Component_means(:, :, t-1), ...
        'Sigma', Component_cov(:, :, :, t-1), ...
        'ComponentProportion', Component_weights(:, :, t-1)), ...
        'Options', statset('MaxIter', 1000), 'RegularizationValue', 1e-6);

    % Storing the parameters
    Component_means(:, :, t) = Gmm.mu;
    Component_cov(:, :, :, t) = Gmm.Sigma;
    Component_weights(:, :, t) = Gmm.ComponentProportion;

    for i = 1:Num_sim % Generating Samples from the GMM
        Chosen_component_idx = find(mnrnd(1, Component_weights(:, :, t)));
        % Use Initial_weights for the current step by choosing component

```

```

    % based on weights
    % Generating Samples from the GMM
    Chosen_mean = Component_means(Chosen_component_idx,:,t);
    Chosen_cov = Component_cov(:,:,Chosen_component_idx,t);
    Sim_returns = mvnrnd(Chosen_mean,Chosen_cov); % Generate return
    Simulated_returns_Gmm1(i,:,t) = Sim_returns.*Adj; % Adjusted stock
    % returns
end

% Calculating VaR and ES for subsequent evaluation days
Simulated_port_ret_Gmm1(:,t) = Simulated_returns_Gmm1(:,:,t)...
    *Stock_weights;

Sorted_port_ret_Gmm1(:,t) = sort(Simulated_port_ret_Gmm1(:,t));

VaR_GMM_alpha1day1(t) = quantile(Sorted_port_ret_Gmm1(:,t), alpha1);
ES_GMM_alpha1day1(t) =...
mean(Sorted_port_ret_Gmm1(Sorted_port_ret_Gmm1(:,t)...
<= VaR_GMM_alpha1day1(t),t));

    VaR_GMM_alpha2day1(t) = quantile(Sorted_port_ret_Gmm1(:,t), alpha2);
    ES_GMM_alpha2day1(t) =...
    mean(Sorted_port_ret_Gmm1(Sorted_port_ret_Gmm1(:,t)...
    <= VaR_GMM_alpha2day1(t),t));
end

%% 5-day VaR and ES

Num_days = 5; % Number of days for evaluating VaR
Simulated_returns_Gmm2 = zeros(Num_sim,Num_stocks,Num_steps-Num_days+1);
% 5-day overlapping returns
Simulated_port_ret_Gmm2 = zeros(Num_sim,Num_steps-Num_days+1);
Sorted_port_ret_Gmm2 = zeros(Num_sim,Num_steps-Num_days+1);

VaR_GMM_alpha1day5 = zeros(1,Num_steps-Num_days+1);
ES_GMM_alpha1day5 = zeros(1,Num_steps-Num_days+1);

VaR_GMM_alpha2day5 = zeros(1,Num_steps-Num_days+1);

```

```

ES_GMM_alpha2day5 = zeros(1,Num_steps-Num_days+1);

% computing 5-day overlapping returns
for r = 1:(Num_steps - Num_days + 1)
    Simulated_returns_Gmm2(:, :, r) = sum(Simulated_returns_Gmm1(:, :, r:r...
    + Num_days - 1), 3);
    Simulated_port_ret_Gmm2(:, r) = Simulated_returns_Gmm2(:, :, r)*...
    Stock_weights;

    Sorted_port_ret_Gmm2(:, r) = sort(Simulated_port_ret_Gmm2(:, r));
    VaR_GMM_alpha1day5(r) = quantile(Sorted_port_ret_Gmm2(:, r), alpha1);
    ES_GMM_alpha1day5(r) = ...
    mean(Sorted_port_ret_Gmm2(Sorted_port_ret_Gmm2(:, r)...
    <= VaR_GMM_alpha1day5(r), r));

    VaR_GMM_alpha2day5(r) = quantile(Sorted_port_ret_Gmm2(:, r), alpha2);
    ES_GMM_alpha2day5(r) = ...
    mean(Sorted_port_ret_Gmm2(Sorted_port_ret_Gmm2(:, r)...
    <= VaR_GMM_alpha2day5(r), r));

end

%% Saving VaR, ES, and Return data to a file
save('VaR1d1_data.mat', 'VaR_GMM_alpha1day1', '-append');
save('ES1d1_data.mat', 'ES_GMM_alpha1day1', '-append');

save('VaR2d1_data.mat', 'VaR_GMM_alpha2day1', '-append');
save('ES2d1_data.mat', 'ES_GMM_alpha2day1', '-append');

save('VaR1d5_data.mat', 'VaR_GMM_alpha1day5', '-append');
save('ES1d5_data.mat', 'ES_GMM_alpha1day5', '-append');

save('VaR2d5_data.mat', 'VaR_GMM_alpha2day5', '-append');
save('ES2d5_data.mat', 'ES_GMM_alpha2day5', '-append');

save('Simulated_port_ret_data1.mat', 'Simulated_port_ret_Gmm1', '-append');
save('Simulated_returns1.mat', 'Simulated_returns_Gmm1', '-append');

```

```
save('Simulated_port_ret_data2.mat', 'Simulated_port_ret_Gmm2', '-append');
save('Simulated_returns2.mat', 'Simulated_returns_Gmm2', '-append');
```

Appendix B

```
% GBM CODE
clc; close all; clear all;
% Trial 1
%% Historical Data
Data = xlsread('Stock_prices_adjusted.xlsx', 'Normal_Abnormal_CashChanges');
Data(:,1) = [];
Data = Data(3523:4109,:); % Data from 01/01/2019 to 31/03/2021
Num_observations = size(Data,1); % Number of observations
Num_stocks = size(Data,2); % Number of Stocks
Stock_names(1:Num_stocks) = ["SPM IM" "ENI IM" "ISP IM" "UCG IM" "G IM"...
    "TEN IM" "TRN IM" "TIT IM" "LDO I" "BPE IM" "SRG IM" "CPR IM"...
    "UNI IM" "AZM IM" "AMP IM"];
Daily_returns = diff(log(Data)); % Calculating daily log returns
Stock_weights = ones(Num_stocks, 1) / Num_stocks; % Equal weights for
% portfolio

%% Descriptive Statistics of return series

return_mean = mean(Daily_returns); % mean of the returns in each column
return_var = var(Daily_returns); % daily variance of the returns
return_correlation = corr(Daily_returns);

Sigma_long = sqrt(var(Daily_returns(1:252,:))); % Standard deviation of
% long period(1 working year)
Sigma_short = sqrt(var(Daily_returns(end-59:end,:))); % Short period
% standard deviation (60 days)
Adj = Sigma_short./Sigma_long;

%% Simulation Parameters, and Stock Price Simulation
% Parameters

mu = return_mean; % Expected returns (mean returns) for each stock
```

```

sigma = sqrt(return_var); % Variance for each stock
T1 = 1;                    % Time horizon (in years)
T2 = 5;
Num_sim = 1000;           % Number of Monte Carlo simulations
alpha1 = 0.01;            % Significance level for VaR
alpha2 = 0.05;

% Simulation
dt1 = 1/1000;             % daily time increment

Num_steps = 1 / dt1;      % Number of time steps

Simulated_stock_prices = zeros(Num_sim, Num_stocks, Num_steps+1);
% Matrix to store simulated prices
Simulated_stock_prices(:, :, 1) = repmat(Data(end, :), Num_sim, 1);
% Initial prices for the stocks---last historical data

rng(0); % To ensure reproducibility
% Generate correlated random returns
A = chol(return_correlation, 'lower'); % decompose correlation matrix into
% the product of a lower triangular matrix and its transpose.

Simulated_returns_GBM1 = zeros(Num_sim, Num_stocks, Num_steps);

Simulated_port_ret_GBM1 = zeros(Num_sim, Num_steps);
Sorted_port_ret_GBM1 = zeros(Num_sim, Num_steps);

VaR_GBM_alpha1day1 = zeros(1, Num_steps);
ES_GBM_alpha1day1 = zeros(1, Num_steps);

VaR_GBM_alpha2day1 = zeros(1, Num_steps);
ES_GBM_alpha2day1 = zeros(1, Num_steps);

```

```

% Simulating stock price paths for subsequent time steps for 1-day VaR
for t = 2:Num_steps + 1

    Random_returns = randn(Num_sim, Num_stocks) * A;

    for i = 1:Num_sim
        Simulated_stock_prices(i,:,t) = Simulated_stock_prices(i,:,t-1)...
            .* exp((mu - 0.5 * sigma.^2) * dt1 + sigma .* sqrt(dt1) .*...
            Random_returns(i,:));
    end
    Simulated_returns_GBM1(:, :, t-1) = log(Simulated_stock_prices(:, :, t) ./...
        Simulated_stock_prices(:, :, t-1)).*Adj;
    Simulated_port_ret_GBM1(:, t-1) = Simulated_returns_GBM1(:, :, t-1)...
        *Stock_weights;

    % Calculating VaR and ES for subsequent evaluation day

    Sorted_port_ret_GBM1(:, t-1) = sort(Simulated_port_ret_GBM1(:, t-1));

    VaR_GBM_alpha1day1(t-1) = quantile(Sorted_port_ret_GBM1(:, t-1), alpha1);
    ES_GBM_alpha1day1(t-1) =...
    mean(Sorted_port_ret_GBM1(Sorted_port_ret_GBM1(:, t-1)...
    <= VaR_GBM_alpha1day1(t-1), t-1));

    VaR_GBM_alpha2day1(t-1) = quantile(Sorted_port_ret_GBM1(:, t-1), alpha2);
    ES_GBM_alpha2day1(t-1) =...
    mean(Sorted_port_ret_GBM1(Sorted_port_ret_GBM1(:, t-1)...
    <= VaR_GBM_alpha2day1(t-1), t-1));
end

%% 5-day VaR and ES

Num_days = 5; % Number of days for evaluating VaR
Simulated_returns_GBM2 = zeros(Num_sim, Num_stocks, Num_steps-Num_days+1);
% 5-day overlapping returns
Simulated_port_ret_GBM2 = zeros(Num_sim, Num_steps-Num_days+1);
Sorted_port_ret_GBM2 = zeros(Num_sim, Num_steps-Num_days+1);

```

```

VaR_GBM_alpha1day5 = zeros(1,Num_steps-Num_days+1);
ES_GBM_alpha1day5 = zeros(1,Num_steps-Num_days+1);

VaR_GBM_alpha2day5 = zeros(1,Num_steps-Num_days+1);
ES_GBM_alpha2day5 = zeros(1,Num_steps-Num_days+1);

% computing 5-day overlapping returns
for r = 1:(Num_steps - Num_days + 1)
    Simulated_returns_GBM2(:, :, r) = sum(Simulated_returns_GBM1(:, :, r:r...
    + Num_days - 1), 3);
    Simulated_port_ret_GBM2(:, r) = Simulated_returns_GBM2(:, :, r)...
    *Stock_weights;

    Sorted_port_ret_GBM2(:, r) = sort(Simulated_port_ret_GBM2(:, r));
    VaR_GBM_alpha1day5(r) = quantile(Sorted_port_ret_GBM2(:, r), alpha1);
    ES_GBM_alpha1day5(r) = ...
    mean(Sorted_port_ret_GBM2(Sorted_port_ret_GBM2(:, r)...
    <= VaR_GBM_alpha1day5(r), r));

    VaR_GBM_alpha2day5(r) = quantile(Sorted_port_ret_GBM2(:, r), alpha2);
    ES_GBM_alpha2day5(r) = ...
    mean(Sorted_port_ret_GBM2(Sorted_port_ret_GBM2(:, r)...
    <= VaR_GBM_alpha2day5(r), r));

end

%%
save('VaR1d1_data.mat', 'VaR_GBM_alpha1day1');
save('ES1d1_data.mat', 'ES_GBM_alpha1day1');

save('VaR2d1_data.mat', 'VaR_GBM_alpha2day1');
save('ES2d1_data.mat', 'ES_GBM_alpha2day1');

save('VaR1d5_data.mat', 'VaR_GBM_alpha1day5');
save('ES1d5_data.mat', 'ES_GBM_alpha1day5');

save('VaR2d5_data.mat', 'VaR_GBM_alpha2day5');
save('ES2d5_data.mat', 'ES_GBM_alpha2day5');

```

```

save('Simulated_port_ret_data1.mat', 'Simulated_port_ret_GBM1');
save('Simulated_returns1.mat', 'Simulated_returns_GBM1');

save('Simulated_port_ret_data2.mat', 'Simulated_port_ret_GBM2');
save('Simulated_returns2.mat', 'Simulated_returns_GBM2');

save('Stock_weights.mat', 'Stock_weights');
save('Stock_names.mat', 'Stock_names');

```

Appendix C

```

% RQMC-GBM CODE
clc; close all; clear all;

%% Historical Data
Data = xlsread('Stock_prices_adjusted.xlsx', 'Normal_Abnormal_CashChanges');
Data(:,1) = [];
Data = Data(3523:4109,:); % Data from 01/01/2019 to 31/03/2021
Num_observations = size(Data,1); % Number of observations
Num_stocks = size(Data,2); % Number of Stocks
Stock_names(1:Num_stocks) = ["SPM IM" "ENI IM" "ISP IM" "UCG IM" "G IM"...
    "TEN IM" "TRN IM" "TIT IM" "LDO I" "BPE IM" "SRG IM" "CPR IM"...
    "UNI IM" "AZM IM" "AMP IM"];
Daily_returns = diff(log(Data)); % Calculating daily log returns
Stock_weights = ones(Num_stocks, 1) / Num_stocks;
% Equal weights for portfolio.

%% Descriptive Statistics of return series

```

```

return_mean = mean(Daily_returns); % mean of the returns in each column
return_var = var(Daily_returns) ; % daily variance of the returns
return_correlation = corr(Daily_returns);

Sigma_long = sqrt(var(Daily_returns(1:252,:))); % Standard deviation of
% long period(1 working year)
Sigma_short = sqrt(var(Daily_returns(end-59:end,:)));% Short period
% standard deviation (60 days)
Adj = Sigma_short./Sigma_long;
%% Simulation Parameters
mu = return_mean; % Expected returns (mean returns) for each stock
sigma = sqrt(return_var); % Variance for each stock
T1 = 1; % Time horizon (in years)
T2 = 5;
Num_sim = 1000; % Number of Monte Carlo simulations
alpha1 = 0.01; % Significance level for VaR
alpha2 = 0.05;

% Simulation
dt1 = 1/1000; % daily time increment

Num_steps = 1 / dt1; % Number of time steps

Simulated_stock_prices1 = zeros(Num_sim, Num_stocks, Num_steps+1);
% Matrix to store simulated prices
Simulated_stock_prices1(:, :, 1) = repmat(Data(end,:), Num_sim, 1);
% Initial prices for the stocks---last historical data

rng(0); % To ensure reproducibility
% Generate correlated random returns
A = chol(return_correlation, 'lower'); % decompose correlation matrix into
% the product of a lower triangular matrix and its transpose.

Simulated_returns_RQMC_GBM1 = zeros(Num_sim,Num_stocks,Num_steps);

Simulated_port_ret_RQMC_GBM1 = zeros(Num_sim,Num_steps);
Sorted_port_ret_RQMC_GBM1 = zeros(Num_sim,Num_steps);

```

```

VaR_RQMC_GBM_alpha1day1 = zeros(1,Num_steps);
ES_RQMC_GBM_alpha1day1 = zeros(1,Num_steps);

VaR_RQMC_GBM_alpha2day1 = zeros(1,Num_steps);
ES_RQMC_GBM_alpha2day1 = zeros(1,Num_steps);

% Create Sobol sequences
sobol_seq = sobolset(Num_stocks); % Create a Sobol sequence set
sobol_seq = scramble(sobol_seq, 'MatousekAffineOwen');
% Scramble Sobol sequences
scrambled_sobol_all1 = net(sobol_seq, Num_sim * Num_steps);

% Generate all Sobol points

% Reshape the Sobol sequence for easy extraction by time step
scrambled_sobol_all1 = reshape(scrambled_sobol_all1, Num_sim, Num_steps,...
    Num_stocks);

%% Simulation Loop for 1-day VaR and ES
for t = 2:Num_steps + 1
    % Generate scrambled Sobol sequences
    scrambled_sobol = scrambled_sobol_all1(:, t-1, :);
    % Generate correlated random returns
    Random_returns = norminv(scrambled_sobol(1:Num_sim,:), 0, 1) * A;

    % Simulating stock prices with adjusted returns
    for i = 1:Num_sim
        Simulated_stock_prices1(i,:,t) = Simulated_stock_prices1(i,:,t-1) ...
            .* exp((mu - 0.5 * sigma.^2) * dt1 + sigma .* sqrt(dt1) .*...
            Random_returns(i,:));
    end

    % Calculate portfolio value and returns
    Simulated_returns_RQMC_GBM1(:, :,t-1) = log(Simulated_stock_prices1...
        (:,:,t)./Simulated_stock_prices1(:, :,t-1)).*Adj;

```

```

Simulated_port_ret_RQMC_GBM1(:,t-1) = ...
    Simulated_returns_RQMC_GBM1(:,t-1)*Stock_weights;

% Calculating VaR and ES
Sorted_port_ret_RQMC_GBM1(:,t-1) = sort(Simulated_port_ret_RQMC_GBM1(:,...
t-1));
VaR_RQMC_GBM_alpha1day1(t-1) =...
quantile(Sorted_port_ret_RQMC_GBM1(:,t-1), alpha1);
ES_RQMC_GBM_alpha1day1(t-1) = mean(Sorted_port_ret_RQMC_GBM1...
(Sorted_port_ret_RQMC_GBM1(:,t-1) <= VaR_RQMC_GBM_alpha1day1(t-1)));

VaR_RQMC_GBM_alpha2day1(t-1) =...
quantile(Sorted_port_ret_RQMC_GBM1(:,t-1), alpha2);
ES_RQMC_GBM_alpha2day1(t-1) = mean(Sorted_port_ret_RQMC_GBM1...
(Sorted_port_ret_RQMC_GBM1(:,t-1) <= VaR_RQMC_GBM_alpha2day1(t-1)));
end

%% 5-day VaR and ES

Num_days = 5; % Number of days for evaluating VaR
Simulated_returns_RQMC_GBM2 = zeros(Num_sim,Num_stocks,Num_steps-Num_days+1);
% 5-day overlapping returns
Simulated_port_ret_RQMC_GBM2 = zeros(Num_sim,Num_steps-Num_days+1);
Sorted_port_ret_RQMC_GBM2 = zeros(Num_sim,Num_steps-Num_days+1);

VaR_RQMC_GBM_alpha1day5 = zeros(1,Num_steps-Num_days+1);
ES_RQMC_GBM_alpha1day5 = zeros(1,Num_steps-Num_days+1);

VaR_RQMC_GBM_alpha2day5 = zeros(1,Num_steps-Num_days+1);
ES_RQMC_GBM_alpha2day5 = zeros(1,Num_steps-Num_days+1);

% computing 5-day overlapping returns
for r = 1:(Num_steps - Num_days + 1)
    Simulated_returns_RQMC_GBM2(:,r) =...
    sum(Simulated_returns_RQMC_GBM1(:,r:r + Num_days - 1),3);
    Simulated_port_ret_RQMC_GBM2(:,r) = Simulated_returns_RQMC_GBM2(:,r)...
    *Stock_weights;

    Sorted_port_ret_RQMC_GBM2(:,r) = sort(Simulated_port_ret_RQMC_GBM2(:,r));

    VaR_RQMC_GBM_alpha1day5(r) =...

```

```

    quantile(Sorted_port_ret_RQMC_GBM2(:,r), alpha1);
    ES_RQMC_GBM_alpha1day5(r) =...
    mean(Sorted_port_ret_RQMC_GBM2(Sorted_port_ret_RQMC_GBM2(:,r)...
    <= VaR_RQMC_GBM_alpha1day5(r),r));

    VaR_RQMC_GBM_alpha2day5(r) =...
    quantile(Sorted_port_ret_RQMC_GBM2(:,r), alpha2);
    ES_RQMC_GBM_alpha2day5(r) =...
    mean(Sorted_port_ret_RQMC_GBM2(Sorted_port_ret_RQMC_GBM2(:,r)...
    <= VaR_RQMC_GBM_alpha2day5(r),r));

end

%% Saving Data

save('VaR1d1_data.mat', 'VaR_RQMC_GBM_alpha1day1', '-append');
save('ES1d1_data.mat', 'ES_RQMC_GBM_alpha1day1', '-append');

save('VaR2d1_data.mat', 'VaR_RQMC_GBM_alpha2day1', '-append');
save('ES2d1_data.mat', 'ES_RQMC_GBM_alpha2day1', '-append');

save('VaR1d5_data.mat', 'VaR_RQMC_GBM_alpha1day5', '-append');
save('ES1d5_data.mat', 'ES_RQMC_GBM_alpha1day5', '-append');

save('VaR2d5_data.mat', 'VaR_RQMC_GBM_alpha2day5', '-append');
save('ES2d5_data.mat', 'ES_RQMC_GBM_alpha2day5', '-append');

save('Simulated_port_ret_data1.mat', 'Simulated_port_ret_RQMC_GBM1', '-append');
save('Simulated_returns1.mat', 'Simulated_returns_RQMC_GBM1', '-append');

save('Simulated_port_ret_data2.mat', 'Simulated_port_ret_RQMC_GBM2', '-append');
save('Simulated_returns2.mat', 'Simulated_returns_RQMC_GBM2', '-append');

```

Appendix D

```
% CODE FOR GRAPHS & FIGURES
clc; close all; clear All;
%% Loading VaR and ES data
load('VaR1d1_data.mat','VaR_GBM_alpha1day1','VaR_GMM_alpha1day1',...
'VaR_RQMC_GBM_alpha1day1');
load('ES1d1_data.mat','ES_GBM_alpha1day1','ES_GMM_alpha1day1',...
'ES_RQMC_GBM_alpha1day1');

load('VaR2d1_data.mat','VaR_GBM_alpha2day1','VaR_GMM_alpha2day1',...
'VaR_RQMC_GBM_alpha2day1');
load('ES2d1_data.mat','ES_GBM_alpha2day1','ES_GMM_alpha2day1',...
'ES_RQMC_GBM_alpha2day1');

load('VaR1d5_data.mat','VaR_GBM_alpha1day5','VaR_GMM_alpha1day5',...
'VaR_RQMC_GBM_alpha1day5');
load('ES1d5_data.mat','ES_GBM_alpha1day5','ES_GMM_alpha1day5',...
'ES_RQMC_GBM_alpha1day5');

load('VaR2d5_data.mat','VaR_GBM_alpha2day5','VaR_GMM_alpha2day5',...
'VaR_RQMC_GBM_alpha2day5');
load('ES2d5_data.mat','ES_GBM_alpha2day5','ES_GMM_alpha2day5',...
'ES_RQMC_GBM_alpha2day5');

%% Graphs for Daily VaR and ES at 99% Confidence
Num_steps = length(VaR_GMM_alpha1day1);
g = 0; % Set g=1 to save figures
% Plotting VaR for GMM and GBM
figure(1);
plot(1:Num_steps, VaR_GMM_alpha1day1, 'r-', 'LineWidth', 1); hold on;
plot(1:Num_steps, VaR_GBM_alpha1day1, 'g--', 'LineWidth', 2); hold on;
plot(1:Num_steps, VaR_RQMC_GBM_alpha1day1, 'b-', 'LineWidth', 2);
title('1-day 99% VaR Comparison for 1000 days: GMM vs GBM vs RQMC-GBM');
xlabel('Days');
ylabel('99% VaR');
legend('GMM VaR', 'GBM VaR', 'RQMC-GBM VaR');
grid on;
```

```

if g==1;
    saveas(gcf, '1-day 99% VaR_Comparison_GMM_vs_GBM_vs_RQMC-GBM.png');
end

% ES Plot
figure(2);
plot(1:Num_steps, ES_GMM_alpha1day1, 'r-', 'LineWidth', 1.5); hold on;
plot(1:Num_steps, ES_GBM_alpha1day1, 'g--', 'LineWidth', 1.5); hold on;
plot(1:Num_steps, ES_RQMC_GBM_alpha1day1, 'b-', 'LineWidth', 1.5);

title('1-day 99% ES Comparison for 1000 days: GMM vs GBM vs RQMC-GBM');
xlabel('Days');
ylabel('ES');
legend('GMM ES', 'GBM ES', 'RQMC-GBM ES');
grid on;

if g==1;
    saveas(gcf, '1-day 99% ES_Comparison_GMM_vs_GBM_vs_RQMC-GBM.png');
end

%% Graphs for Daily VaR and ES at 95% Confidence
figure(3);
plot(1:Num_steps, VaR_GMM_alpha2day1, 'r-', 'LineWidth', 1); hold on;
plot(1:Num_steps, VaR_GBM_alpha2day1, 'g--', 'LineWidth', 2); hold on;
plot(1:Num_steps, VaR_RQMC_GBM_alpha2day1, 'b-', 'LineWidth', 2);
title('1-day 95% VaR Comparison for 1000 days: GMM vs GBM vs RQMC-GBM');
xlabel('Days');
ylabel('95% VaR');
legend('GMM VaR', 'GBM VaR', 'RQMC-GBM VaR');
grid on;

if g==1;
    saveas(gcf, '1-day 95% VaR_Comparison_GMM_vs_GBM_vs_RQMC-GBM.png');
end

% ES Plot
figure(4);
plot(1:Num_steps, ES_GMM_alpha2day1, 'r-', 'LineWidth', 1.5); hold on;
plot(1:Num_steps, ES_GBM_alpha2day1, 'g--', 'LineWidth', 1.5); hold on;

```

```

plot(1:Num_steps, ES_RQMC_GBM_alpha2day1, 'b-', 'LineWidth', 1.5);

title('1-day 95% ES Comparison for 1000 days: GMM vs GBM vs RQMC-GBM');
xlabel('Days');
ylabel('ES');
legend('GMM ES', 'GBM ES', 'RQMC-GBM ES');
grid on;

if g==1;
    saveas(gcf, '1-day 95% ES_Comparison_GMM_vs_GBM_vs_RQMC-GBM.png');
end

%% Graphs for 5-day VaR and ES at 99% Confidence

% Plotting VaR for GMM and GBM

Num_steps1 = length(VaR_GMM_alpha1day5);
figure(5);
plot(1:Num_steps1, VaR_GMM_alpha1day5, 'r-', 'LineWidth', 1); hold on;
plot(1:Num_steps1, VaR_GBM_alpha1day5, 'g--', 'LineWidth', 2); hold on;
plot(1:Num_steps1, VaR_RQMC_GBM_alpha1day5, 'b-', 'LineWidth', 2);

title('5-day 99% VaR Comparison: GMM vs GBM vs RQMC-GBM');
xlabel('Days');
ylabel('99% VaR');
legend('GMM VaR', 'GBM VaR', 'RQMC-GBM VaR');
grid on;

if g==1;
    saveas(gcf, '5-day 99% VaR_Comparison_GMM_vs_GBM_vs_RQMC-GBM.png');
end

% ES Plot
figure(6);
plot(1:Num_steps1, ES_GMM_alpha1day5, 'r-', 'LineWidth', 1.5); hold on;
plot(1:Num_steps1, ES_GBM_alpha1day5, 'g--', 'LineWidth', 1.5); hold on;
plot(1:Num_steps1, ES_RQMC_GBM_alpha1day5, 'b-', 'LineWidth', 1.5);

```

```

title('5-day 99% ES Comparison for 1000 days: GMM vs GBM vs RQMC-GBM');
xlabel('Days');
ylabel('ES');
legend('GMM ES', 'GBM ES', 'RQMC-GBM ES');
grid on;

if g==1;
    saveas(gcf, '5-day 99% ES_Comparison_GMM_vs_GBM_vs_RQMC-GBM.png');
end

%% Graphs for 5-day VaR and ES at 95% Confidence
figure(7);
plot(1:Num_steps1, VaR_GMM_alpha2day5, 'r-', 'LineWidth', 1); hold on;
plot(1:Num_steps1, VaR_GBM_alpha2day5, 'g--', 'LineWidth', 2); hold on;
plot(1:Num_steps1, VaR_RQMC_GBM_alpha2day5, 'b-', 'LineWidth', 2);

title('5-day 95% VaR Comparison for 1000 days: GMM vs GBM vs RQMC-GBM');
xlabel('Days');
ylabel('95% VaR');
legend('GMM VaR', 'GBM VaR', 'RQMC-GBM VaR');
grid on;

if g==1;
    saveas(gcf, '5-day 95% VaR_Comparison_GMM_vs_GBM_vs_RQMC-GBM.png');
end

% ES Plot
figure(8);
plot(1:Num_steps1, ES_GMM_alpha2day5, 'r-', 'LineWidth', 1.5); hold on;
plot(1:Num_steps1, ES_GBM_alpha2day5, 'g--', 'LineWidth', 1.5); hold on;
plot(1:Num_steps1, ES_RQMC_GBM_alpha2day5, 'b-', 'LineWidth', 1.5);

title('5-day 95% ES Comparison for 1000 days: GMM vs GBM vs RQMC-GBM');
xlabel('Days');
ylabel('ES');
legend('GMM ES', 'GBM ES', 'RQMC-GBM ES');
grid on;

```

```

if g==1;
    saveas(gcf, '5-day 95% ES_Comparison_GMM_vs_GBM_vs_RQMC-GBM.png');
end

%% Graph for Simulated portfolio returns.
load('Simulated_port_ret_data1.mat','Simulated_port_ret_GBM1',...
    'Simulated_port_ret_Gmm1','Simulated_port_ret_RQMC_GBM1');

Num_sim = size(Simulated_port_ret_Gmm1,1);
% Calculating mean portfolio returns over all simulations for each time step
% mean_port_ret_GBM = mean(Simulated_port_ret_GBM, 1);
% Mean over simulations
% mean_port_ret_Gmm = mean(Simulated_port_ret_Gmm, 1);
% mean_port_ret_RQMC_GBM = mean(Simulated_port_ret_RQMC_GBM, 1);

% Plotting the mean returns for each method
figure(9);
plot(1:Num_sim, Simulated_port_ret_GBM1(:,1), 'g-', 'LineWidth', 1.5); hold on;
plot(1:Num_sim, Simulated_port_ret_Gmm1(:,1), 'r-', 'LineWidth', 1.5); hold on;
plot(1:Num_sim, Simulated_port_ret_RQMC_GBM1(:,1), 'b-.', 'LineWidth', 1.5);

title('Comparison of Simulated Portfolio Returns: GBM vs GMM vs RQMC GBM');
xlabel('Number of Simulations');
ylabel('Portfolio Return');
legend('GBM', 'GMM', 'RQMC GBM');
grid on;

if g==1;
    saveas(gcf,...
        'Simulated Portfolio Return Comparison_GMM_vs_GBM_vs_RQMC-GBM.png');
end

%% Scatter plots of historical data(2) and GMM Clustering with 3 components
load('Daily_returns_data.mat','Daily_returns');

Stock_names(1:15) = ["SPM IM" "ENI IM" "ISP IM" "UCG IM" "G IM"...
    "TEN IM" "TRN IM" "TIT IM" "LDO I" "BPE IM" "SRG IM" "CPR IM" "UNI IM"...
    "AZM IM" "AMP IM"];

for t = 2:4

```

```

X = Daily_returns(:,[1,t]);
% Scatter plot of historical data
figure(10);
subplot(2,3,t-1);
scatter(X(:,1),X(:,2),10,'o')
title('Historical Data')
grid on;

% fitting a Gmm

Num_components = 3;

[n,p] = size(X);
options = statset('MaxIter',1000);

gmfit = fitgmdist(X,Num_components,'CovarianceType','diagonal',...
    'SharedCovariance',false,'Options',options);
clusterind = cluster(gmfit,X);

% plotting the results
x1 = linspace(min(X(:,1))-2, max(X(:,1))+2, 500);
x2 = linspace(min(X(:,2))-2, max(X(:,2))+2, 500);
[x1grid,x2grid] = meshgrid(x1,x2);
X0 = [x1grid(:) x2grid(:)];

mahalDist = mahal(gmfit,X0);

subplot(2,3,t+2);
h1=gscatter(X(:,1),X(:,2),clusterind);
hold on
plot(gmfit.mu(:,1),gmfit.mu(:,2),'kx','LineWidth',2,'MarkerSize',10)

threshold = sqrt(chi2inv(0.99,2));
for m = 1:Num_components
    idx = mahalDist(:,m)<=threshold;
    Color = h1(m).Color;
    plot(X0(idx,1),X0(idx,2),'.','Color',Color,'MarkerSize',1);
end

```

```

legend off;
title(Stock_names(t));
grid on;

if g==1;
    saveas(gcf,...
        'Scatter plots of historical data(2) and GMM Clustering with
        3 components.png');
end
end
%% Scatter plots of historical data(3-11) and GMM Clustering with 3 components

for t = 5:13;
    X = Daily_returns(:,[1,t]);
    % Scatter plot of historical data
    figure(11);

% fitting a Gmm

    Num_components = 3;

    [n,p] = size(X);
    options = statset('MaxIter',1000);

    gmfit = fitgmdist(X,Num_components,'CovarianceType','diagonal',...
        'SharedCovariance',false,'Options',options);
    clusterind = cluster(gmfit,X);

% plotting the results
    x1 = linspace(min(X(:,1))-2, max(X(:,1))+2, 500);
    x2 = linspace(min(X(:,2))-2, max(X(:,2))+2, 500);
    [x1grid,x2grid] = meshgrid(x1,x2);
    X0 = [x1grid(:) x2grid(:)];

    mahalDist = mahal(gmfit,X0);

    subplot(3,3,t-4);
    h1=gscatter(X(:,1),X(:,2),clusterind);
    hold on
    plot(gmfit.mu(:,1),gmfit.mu(:,2),'kx','LineWidth',2,'MarkerSize',10)

```

```

threshold = sqrt(chi2inv(0.99,2));
for m = 1:Num_components
    idx = mahalDist(:,m)<=threshold;
    Color = h1(m).Color;
    plot(X0(idx,1),X0(idx,2),'.','Color',Color,'MarkerSize',1);
end
legend off;
title(Stock_names(t));
grid on;

if g==1;
    saveas(gcf,...
        'Scatter plots of historical data(3) and GMM Clustering with
        3 components.png');
end
end

%% Plotting the distribution of daily returns and overlaying normal and GMM

for t = 1:9;
    X = Daily_returns(:,t);

    % Plotting histogram
    figure(12);
    subplot(3,3,t)
    histogram(X, 50, 'Normalization', 'pdf', 'FaceAlpha', 0.5,...
        'EdgeColor', 'none');

    hold on;

    % Overlaying Normal Distribution
    mu_norm = mean(X);
    sigma_norm = std(X);

    x_values = linspace(min(X), max(X), 500);
    normal_pdf = normpdf(x_values, mu_norm, sigma_norm);

    plot(x_values, normal_pdf, 'r-', 'LineWidth', 2);

    % Overlaying GMM

```

```

Num_components = 3;

[n,p] = size(X);
options = statset('MaxIter',1000);

gmfit = fitgmdist(X,Num_components,'CovarianceType','diagonal',...
    'SharedCovariance',false,'Options',options,...
    'RegularizationValue', 1e-6);
clusterind = cluster(gmfit,X);

% Evaluating the GMM PDF over the same range
gmm_pdf = pdf(gmfit, x_values');

plot(x_values, gmm_pdf, 'b', 'LineWidth', 2);

xlabel('Daily Returns');
ylabel('Probability');

legend(Stock_names(t), 'Normal', 'GMM','show', 'FontSize', 5);

grid on;
end
if g==1;
    saveas(gcf, 'Frequency_Distribution_with_Normal_and_GMM.png');
end

%% Plotting the distribution of simulated daily returns ...
% and overlaying normal and GMM.

mu = permute(squeeze(mean(Simulated_returns_Gmm1,1)), [2, 1]);

for t = 1:9;
    X = mu(:,t);
    % Plotting histogram
    figure(13);
    subplot(3,3,t)
    histogram(X, 50, 'Normalization', 'pdf', 'FaceAlpha', 0.5,...
        'EdgeColor', 'none');

```

```

hold on;

% Overlaying Normal Distribution
mu_norm = mean(X);
sigma_norm = std(X);

x_values = linspace(min(X), max(X), 500);
normal_pdf = normpdf(x_values, mu_norm, sigma_norm);

plot(x_values, normal_pdf, 'r-', 'LineWidth', 2);

% Overlaying GMM
Num_components = 3;

[n,p] = size(X);
options = statset('MaxIter',1000);

gmfit = fitgmdist(X,Num_components,'CovarianceType','diagonal',...
    'SharedCovariance',false,'Options',options,...
    'RegularizationValue', 1e-6);
clusterind = cluster(gmfit,X);

% Evaluating the GMM PDF over the same range
gmm_pdf = pdf(gmfit, x_values');

plot(x_values, gmm_pdf, 'b', 'LineWidth', 2);

xlabel('Daily Returns');
ylabel('Probability');
%legend('show', 'FontSize', 10);
legend(Stock_names(t), 'Normal', 'GMM','show');

grid on;
end
if g==1;
    saveas(gcf, 'Frequency_Distribution_with_Normal_and_GMM.png');
end

```

Appendix E

```
% CODE FOR TESTS & TABLES
clc; clear all; close all;
%% Descriptive Statistics
% Empirical_return_dist = sort(Empirical_portfolio_return);
% Empirical Distribution of returns
% Empirical_Var = quantile(Empirical_return_dist, alpha);
% Rmse = sqrt(mean(( Empirical_return_dist- Sorted_returns).^2));

g=0;
load('Daily_returns_data.mat','Daily_returns')
load('Simulated_returns1.mat','Simulated_returns_GBM1',...
     'Simulated_returns_Gmm1','Simulated_returns_RQMC_GBM1');
load('Stock_names.mat','Stock_names');
return_mean = mean(Daily_returns); % mean of the returns in each column
return_mode = mode(Daily_returns);
return_median = median(Daily_returns);
return_one_quantile = quantile(Daily_returns,0.01);
return_var = var(Daily_returns) ; % daily variance of the returns
return_skew = skewness(Daily_returns); % skewness of returns
return_kurt = kurtosis(Daily_returns); % kurtosis of returns
return_max = max(Daily_returns);
return_min = min(Daily_returns);
return_correlation = corr(Daily_returns);
Indicatorname = ["Mean","Mode","Median","Quantiles","Variance",...
                "Skewness","Kurtosis","Maximum","Minimum"];
Table_stats = table(return_mean', return_mode', return_median',...
                   return_one_quantile', return_var', return_skew', return_kurt',...
                   return_max', return_min', 'VariableNames', Indicatorname,...
                   'RowNames', Stock_names);
writetable(Table_stats,"Table_statistics.xlsx",'WriteRowNames',true)

fig = figure('Position', [100, 100, 800, 400]);

uitable('Data', table2cell(Table_stats),...
       'ColumnName', Table_stats.Properties.VariableNames, ...
       'RowName', Table_stats.Properties.RowNames, 'Units',...
       'Normalized', 'Position', [0, 0, 1, 1]);
```

```

if g ==1;
    saveas(fig, 'Descriptive Statistics.png');
end

%% RMSE Goodness-of-fit Test
Num_stocks = size(Daily_returns,2);
N = size(Daily_returns, 1); % Number of actual observations
RMSE_GBM = zeros(1, Num_stocks); % Store RMSE for each day
RMSE_RQMC_GBM = zeros(1, Num_stocks);
RMSE_Gmm = zeros(1, Num_stocks);

for t = 1:Num_stocks
    % Simulated returns (mean over simulations)
    Simulated_mean_returns_GBM = squeeze(mean(Simulated_returns_GBM1...
    (:,t,:), 1));
    Simulated_mean_returns_RQMC_GBM = ...
    squeeze(mean(Simulated_returns_RQMC_GBM1(:,t,:), 1));
    Simulated_mean_returns_Gmm = squeeze(mean(Simulated_returns_Gmm1...
    (:,t,:), 1));

    % Actual returns
    Actual_returns = Daily_returns(:, t);

    % Compute RMSE
    RMSE_GBM(t) = sqrt(mean((Simulated_mean_returns_GBM(1:N) -...
    Actual_returns).^2));
    RMSE_RQMC_GBM(t) = sqrt(mean((Simulated_mean_returns_RQMC_GBM...
    (1:N) - Actual_returns).^2));
    RMSE_Gmm(t) = sqrt(mean((Simulated_mean_returns_Gmm...
    (1:N) - Actual_returns).^2));
end

%% RMSE Table
Vertical = ["GMM", "GBM", "RQMC_GBM"];
Table_RMSE = table(RMSE_Gmm', RMSE_GBM', RMSE_RQMC_GBM',...
    'VariableNames',Vertical, 'RowNames', Stock_names);
writetable(Table_RMSE,"Table_RMSE.xlsx",'WriteRowNames',true);

```

```

fig = figure('Position', [100, 100, 800, 400]);

uitable('Data', table2cell(Table_RMSE), 'ColumnName',...
Table_RMSE.Properties.VariableNames, ...'RowName',...
Table_RMSE.Properties.RowNames,
'Units','Normalized', 'Position', [0, 0, 1, 1]);

if g ==1;
    saveas(fig, 'RMSE Test Results.png');
end

%% Christoffersen VaR validity test

load('Stock_weights.mat','Stock_weights');
load('VaR_data.mat','VaR_GBM','VaR_GMM','VaR_RQMC_GBM');

% Calculating Portfolio Returns
Empirical_portfolio_returns = Daily_returns * Stock_weights;

% Significance level for VaR
alpha = 0.01;

% Initializing variables
LRuc_Gmm = 0;    % Likelihood ratio for unconditional coverage
LRind_Gmm = 0;  % Likelihood ratio for independence
pValUC_Gmm = 0; % p-value for unconditional coverage test
pValInd_Gmm = 0; % p-value for independence test

% Identifying VaR exceptions
Exceptions_Gmm = Empirical_portfolio_returns < VaR_GMM(1);
% Binary vector of exceptions
Sum_Gmm = sum(Exceptions_Gmm);
% Total number of exceptions

% Step 3.1: Unconditional Coverage Test (LRuc)
p_hat = Sum_Gmm / N;

```

```

% Observed exception rate
LRuc_Gmm = -2 * (log((1 - alpha)^(N - Sum_Gmm) * alpha^Sum_Gmm)...
- log((1 - p_hat)^(N - Sum_Gmm) * p_hat^Sum_Gmm));
pValUC_Gmm = 1 - chi2cdf(LRuc_Gmm, 1);
% p-value for unconditional coverage

% Step 3.2: Independence Test (LRind)
% Count transitions between exception (1) and non-exception (0)
n00 = 0; n01 = 0; n10 = 0; n11 = 0;
for t = 2:N
    if Exceptions_Gmm(t-1) == 0 && Exceptions_Gmm(t) == 0
        n00 = n00 + 1;
    elseif Exceptions_Gmm(t-1) == 0 && Exceptions_Gmm(t) == 1
        n01 = n01 + 1;
    elseif Exceptions_Gmm(t-1) == 1 && Exceptions_Gmm(t) == 0
        n10 = n10 + 1;
    elseif Exceptions_Gmm(t-1) == 1 && Exceptions_Gmm(t) == 1
        n11 = n11 + 1;
    end
end

% Transition probabilities
pi01 = n01 / (n00 + n01);
% Probability of transitioning from no exception to exception
pi11 = n11 / (n10 + n11);
% Probability of staying in the exception state
pi2 = (n01 + n11) / (n00 + n01 + n10 + n11);
% Overall probability of an exception

% Likelihood ratio test for independence
LRind_Gmm = -2 * (log((1 - pi2)^(n00 + n10) * pi2^(n01 + n11)) - ...
(log((1 - pi01)^n00 * pi01^n01 * (1 - pi11)^n10 * pi11^n11)));
pValInd_Gmm = 1 - chi2cdf(LRind_Gmm, 1); % p-value for independence

%%
% Initializing variables
LRuc_GBM = 0; % Likelihood ratio for unconditional coverage
LRind_GBM = 0; % Likelihood ratio for independence
pValUC_GBM = 0; % p-value for unconditional coverage test
pValInd_GBM = 0; % p-value for independence test

```

```

Exceptions_GBM = Empirical_portfolio_returns < VaR_GBM(1);
% Binary vector of exceptions
Sum_GBM = sum(Exceptions_GBM); % Total number of exceptions

% Step 3.1: Unconditional Coverage Test (LRuc)
p_hat = Sum_GBM / N; % Observed exception rate
LRuc_GBM = -2 * (log((1 - alpha)^(N - Sum_GBM) * alpha^Sum_GBM) - ...
log((1 - p_hat)^(N - Sum_GBM) * p_hat^Sum_GBM));
pValUC_GBM = 1 - chi2cdf(LRuc_GBM, 1); % p-value for unconditional coverage

% Step 3.2: Independence Test (LRind)
% Count transitions between exception (1) and non-exception (0)
n00 = 0; n01 = 0; n10 = 0; n11 = 0;
for t = 2:N
    if Exceptions_GBM(t-1) == 0 && Exceptions_GBM(t) == 0
        n00 = n00 + 1;
    elseif Exceptions_GBM(t-1) == 0 && Exceptions_GBM(t) == 1
        n01 = n01 + 1;
    elseif Exceptions_GBM(t-1) == 1 && Exceptions_GBM(t) == 0
        n10 = n10 + 1;
    elseif Exceptions_GBM(t-1) == 1 && Exceptions_GBM(t) == 1
        n11 = n11 + 1;
    end
end

% Transition probabilities
pi01 = n01 / (n00 + n01);
% Probability of transitioning from no exception to exception
pi11 = n11 / (n10 + n11);
% Probability of staying in the exception state
pi2 = (n01 + n11) / (n00 + n01 + n10 + n11);
% Overall probability of an exception

% Likelihood ratio test for independence
LRind_GBM = -2 * (log((1 - pi2)^(n00 + n10) * pi2^(n01 + n11)) - ...
(log((1 - pi01)^n00 * pi01^n01 * (1 - pi11)^n10 * pi11^n11)));
pValInd_GBM = 1 - chi2cdf(LRind_GBM, 1); % p-value for independence

%%

```

```

% Initializing variables
LRuc_RQMC_GBM = 0; % Likelihood ratio for unconditional coverage
LRind_RQMC_GBM = 0; % Likelihood ratio for independence
pValUC_RQMC_GBM = 0; % p-value for unconditional coverage test
pValInd_RQMC_GBM = 0; % p-value for independence test
Exceptions_RQMC_GBM = Empirical_portfolio_returns < VaR_RQMC_GBM(1);
% Binary vector of exceptions
Sum_RQMC_GBM = sum(Exceptions_RQMC_GBM); % Total number of exceptions

% Step 3.1: Unconditional Coverage Test (LRuc)
p_hat = Sum_RQMC_GBM / N; % Observed exception rate
LRuc_RQMC_GBM = -2 * (log((1 - alpha)^...
(N - Sum_RQMC_GBM) * alpha^Sum_RQMC_GBM) -...
log((1 - p_hat)^(N - Sum_RQMC_GBM) * p_hat^Sum_RQMC_GBM));
pValUC_RQMC_GBM = 1 - chi2cdf(LRuc_RQMC_GBM, 1);
% p-value for unconditional coverage

% Step 3.2: Independence Test (LRind)
% Count transitions between exception (1) and non-exception (0)
n00 = 0; n01 = 0; n10 = 0; n11 = 0;
for t = 2:N
    if Exceptions_RQMC_GBM(t-1) == 0 && Exceptions_RQMC_GBM(t) == 0
        n00 = n00 + 1;
    elseif Exceptions_RQMC_GBM(t-1) == 0 && Exceptions_RQMC_GBM(t) == 1
        n01 = n01 + 1;
    elseif Exceptions_RQMC_GBM(t-1) == 1 && Exceptions_RQMC_GBM(t) == 0
        n10 = n10 + 1;
    elseif Exceptions_RQMC_GBM(t-1) == 1 && Exceptions_RQMC_GBM(t) == 1
        n11 = n11 + 1;
    end
end

% Transition probabilities
pi01 = n01 / (n00 + n01);
% Probability of transitioning from no exception to exception
pi11 = n11 / (n10 + n11);
% Probability of staying in the exception state
pi2 = (n01 + n11) / (n00 + n01 + n10 + n11);
% Overall probability of an exception

% Likelihood ratio test for independence

```

```

LRind_RQMC_GBM = -2 * (log((1 - pi2)^(n00 + n10) * pi2^(n01 + n11)) - ...
    (log((1 - pi01)^(n00 * pi01^n01 * (1 - pi11)^n10 * pi11^n11)));
pValInd_RQMC_GBM = 1 - chi2cdf(LRind_RQMC_GBM, 1);
% p-value for independence

%% Table
UC = [LRuc_Gmm, LRuc_GBM, LRuc_RQMC_GBM];
p_value_uc = [pValUC_Gmm, pValUC_GBM, pValUC_RQMC_GBM];
IND = [LRind_Gmm, LRind_GBM, LRind_RQMC_GBM];
p_Value_ind = [pValInd_Gmm, pValInd_GBM, pValInd_RQMC_GBM];

Verticalnames = ["GMM","GBM","RQMC_GBM"];
Horizontalnames = ["Unconditional Coverage","p-value_uc",...
    "Independence","p-value_ind"];
Table_CTest = table(UC', p_value_uc', IND',p_Value_ind',...
    'VariableNames', Horizontalnames, 'RowNames', Verticalnames);
writetable(Table_CTest,"Table_CTest.xlsx",'WriteRowNames',true)

fig = figure('Position', [100, 100, 800, 400]);
uitable('Data', table2cell(Table_CTest), 'ColumnName',...
    Table_CTest.Properties.VariableNames, ...
    'RowName',Table_CTest.Properties.RowNames,...
    'Units', 'Normalized', 'Position', [0, 0, 1, 1]);

if g ==1;
    saveas(fig, 'Christoffersen Test Results.png');
end

```

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