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Portfolio rebalancing: comparing  
naive and classical strategies.

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# Thanks to.....

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# INTRODUCTION

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Today every investor who faces a new investment can choose among a large variety of financial techniques and strategies available starting from the last century; from simple and easy strategies to more complex and sophisticated strategies. The models are continuously renewing themselves, and the literature is every day implementing new models with the aim to create a portfolio model able to satisfy every economic agent. There are many different kinds of strategies which are capable to take in account the preferences of each investor, or models based on an equal fractioning of the capital, or models highly sophisticated based on the computation of the particular moments of the securities. A simple question arises: which model is the *best*? This question is in the mind of many investors and they have to face it every time they are ready to enter in an investment choice. This answer is not easy. Every model has got its pros and cons, and the literature has not yet proved or given any certain answer, leaving the final choice to every different investor.

One of the main common limits of the models present in the literature today is related to the time-horizon of the strategy built. In fact, in many of the models built till today, starting from the first optimization model of Harry Markowitz (1952), the strategy adopted by the investor is still not able to enter in a series of transactions phases, but it simply consists in three steps: firstly buy the initial assets, secondly hold them till the maturity of the investment, and finally collect the profits (or maybe losses) generated. In this way the strategy is *passive* in front of the market movements and it is not able to adjust the investments initially done; the investor just observes till the maturity of the investment. It is lacking of a system of revision and rebalancing, that could help the investment to become actively managed by the investor during the time

horizon and no more focused on the single-period time horizon, but a multi-periodic investment.

These two open questions that are today present in the literature, gave us the idea to investigate whether simple and naive strategies can be considered to be competitive with respect to a more traditional, sophisticated and consolidated one in a system of rebalancing portfolios. This was done through the development of a test that took inspiration from a previous effort done in the literature by different authors, but with some new modifications in order to take in considerations the critiques in which the original experiment incurred. In this way it has been possible to assess the different model performances, and check how the different strategies behaved. The thesis is structured in the following way:

- In the first chapter the traditional financial models usually used by the investors are presented, explaining the pros and cons of the models and the importance of the diversification principle. In particular have been analyzed the traditional “Mean Variance” optimization portfolio of Harry Markowitz, and two implementations that try overcome its weak point; the “Minimum Variance” model and the “1/N” model, two naive and simple strategies. The description of the portfolio models is integrated with an investment research that showed how a large part of investors usually allocates its wealth.
- In the second chapter, the two literature articles that gave inspiration to the thesis are reported; the article “Optimal versus naive diversification: how inefficient is the 1/N portfolio strategy?”, written by Victor DeMiguel, Lorenzo Garlappi and Roman Uppal , that showed through an empirical experiment the best performance of the 1/N model over the other strategies through a system of rebalancing portfolios; on the other side, it is reported the article “In defense of optimization: the fallacy of 1/N”, written by Kritzman,

Page and Turkington, where the authors argued that the result of the previous article was not reliable because of the estimation criteria adopted. In addition to this, it has been presented a psychological approach to the portfolio investment choice by the psychologist Gerd Gigerenzer and a deeper analysis of the behavior of stocks and portfolios with low risks in the markets.

- In the third chapter, it has been presented the application of the new the experiment. It was developed through the use of new estimation windows of 120, 150 180 and 210 months respect to the original experiment, and three financial portfolio models considered were: “the 1/N, the Mean Variance and the Minimum Variance” model. The experiment is accompanied with the “Matlab” code developed for this particular experiment. Follow the experiment comments and conclusions.
- In the fourth chapter they are exposed the conclusions of the thesis, trying to give an interpretation to portfolio models considered in the experiment according to their performances results, and suggesting a possible direction to follow for the future models.

# CHAPTER 1

## “PORTFOLIO STRATEGIES”

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*"Investing should be more like watching paint dry or watching grass grow.  
If you want excitement, take \$800 and go to Las Vegas."*

Paul Samuelson

### **1.1 Investing with a strategy**

Maybe every day we are investors without even realizing it. Have you ever thought it? Also when you are doing some shopping, you are ignoring that in your mind you are adopting a strategy. Maybe you are taking in consideration your preferences (through optimization model), while at the same time respecting some budget constraints (sum of the asset weights must be equal to 1 or no short-sale is allowed), or maybe you are just dividing your capital in equal fractions (1/N strategy), or keeping some fixed proportions in your purchases (value weighted portfolio). Making a good and successful investment is the most important prerogative for an investor. It is a choice that involves many aspects: strategies, personal skills, experience, risk aversion and emotions. When you decide to invest money, you should use the best solution to achieve your success. This is why an investor should be accurate to adopt the strategy that better represents his choices.

Every investor who is willing to invest in financial markets is called to create a financial portfolio constituted by assets to select. Today different studies have conceived many models that can be used by the investor, like the “Mean Variance optimization”, the “Minimum Variance” model, the “Value weighted portfolio”, “Bayes-Stein shrinkage portfolio”, and many other different models. Each of them applies a different

strategy, and has got its strong and weak points, but it is yet not possible to state that there is a perfect model the investors can choose. It is up to the investor to decide which one is better for him. These strategies are not general academic solutions without connection to modern financial practice.

## **1.2 The diversification principle**

One of the more important aspects of the investment policy is the *diversification principle*. There is an anecdote to give an idea of what we are talking about:

*“In ancient times, the chinses merchants create a method to reduce risks in their trades. To avoid that their shipment were attacked by pirates, they divided them into different ships. In this way, if a ship was assaulted or shipwrecked, the other ships could arrive to the other destinations. There were more chances to conclude the trade.”*

(By Unknown)

According to the diversification principle, when you are investing in an asset or in an activity, your investment can generate a profit but at the same time it can make a loss. In this way the investor is completely exposed to the result of the strategy. But imagine now that the investor decides to invest in more than one activity, like two activities and that they do not belong to same economy branch. When both the 2 two activities are going well the investor can just be happy of his success; but imagine now that something happens in the market that affects the first activity (for example the investor invested in a transport company and suddenly the oil price increases) and you start suffering a loss. If at the time of the investment the investor bought a second activity related to a different market (like a technological company), he probably will not be affected by that market event that caused the losses of the first

investment and the second investment will continue to generate profits. In this way diversification can protect the investor from risks, but it cannot completely remove them. You can manage risk. It is sufficient to invest in more than one asset for example if you are investing in portfolios, buying a number like 20 assets according to some researches. Statistically speaking, the diversification highlights the importance of the correlation between the return of the assets. The correlation is expressed by the formula:

$$\rho_{xy} = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$$

where  $\sigma_{xy}$  is the covariance of the two variables (in this case two assets), while  $\sigma_x$  and  $\sigma_y$  are the respective standard deviations. This index goes from -1 to +1. When the correlation is negative, this means that the two considered variables go in two opposite directions; when the correlation is positive they go in the same direction. When the index is close to zero, they are uncorrelated. So the investor should look for negative correlation if he wants to diversify. But how much the correlation is important?

The correlation can be useful till a certain point. There is a part of the risk, the unsystematic risk, that is specific to each company/asset that can be eliminated through the use of the diversification (for this is useful to look for the proper correlation); but there is a part of the risk that cannot be eliminated, that is the systematic risk. This kind of risk is not a specific risk of the security, but is a part of the risk present in the market taken as whole. This risk is known also as “undiversifiable” risk.

### 1.3 Modern Portfolio Theory



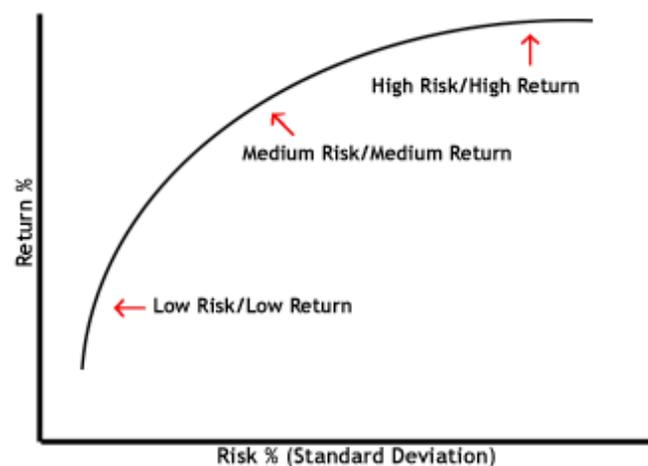
**Figure 1:** Harry Markowitz.

The founder of this branch of economy, relative to the new techniques and portfolio models, is considered Harry Markowitz. In 1952 he came with some important results and a new theory; the “*modern portfolio theory*” (1952). The theory he formulated, that today is still used by some financial banks, stated a new practical financial model that in that time opened a new era of the financial literature that searches for the best financial portfolio: he gave birth to the “Mean-Variance portfolio”. This model has the aim to help the investor to select the optimal portfolio that is able to generate the maximum return for the lowest possible value of risk, according to the risk aversion of the investor. Harry Markowitz demonstrated that it is possible to find the optimal portfolio through a statistical and mathematical approach. First of all it is important to define the main hypothesis of the environment in which the model was born:

- *Frictionless market assumption*: it is a kind of market where all the transaction costs disappear, and there is no taxation on the capital gain;
- *price taker assumption*: the investors are not able to affect the price of the goods through their buying and selling;

- *no institutional restriction assumption*: this means that the regulator cannot impose any restriction on the investments (ex: short-selling is allowed).

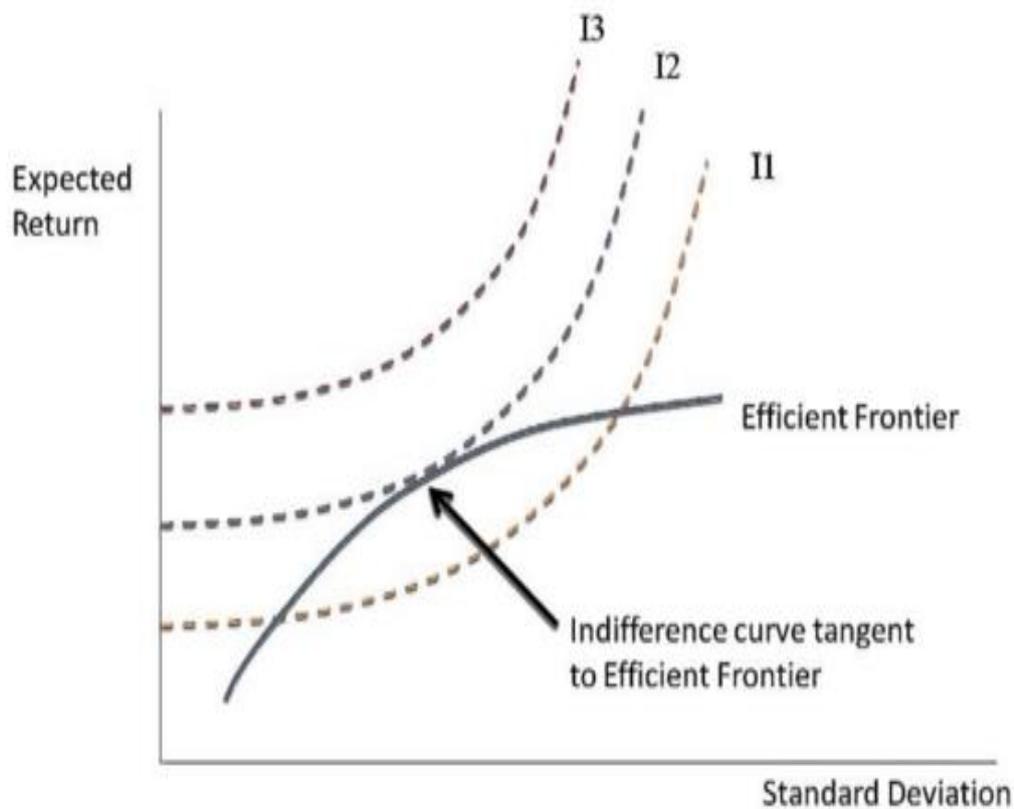
The content of the model presented was quite new for that time and it substantially relies on the principle of diversification and on the moments of the assets considered (the expected return, the variance and the correlation). In particular, the standard deviation is the moment that is able to describe the risk of each asset and of the portfolio, and it is seen as something undesirable. The diversification looks for the selection of assets in a portfolio such that the assets have low positive correlation, with the aim to minimize the portfolio variance without renouncing the expected return. Combining all the available assets, it is possible to form different portfolios, each one of them is characterized by a certain expected return and a certain standard deviation. Among these portfolios, it is possible to identify a set of efficient portfolios; they are a combination of assets that are able to give the maximum expected return for a given level of risk. In this way for each level of risk, there is an efficient portfolio. This set takes the name of “efficient frontier”; it is an upward slowing curve, with the shape of a hyperbola, which represents all the efficient portfolios in which the investor can invest *efficiently*.



**Figure 2:** The efficient frontier

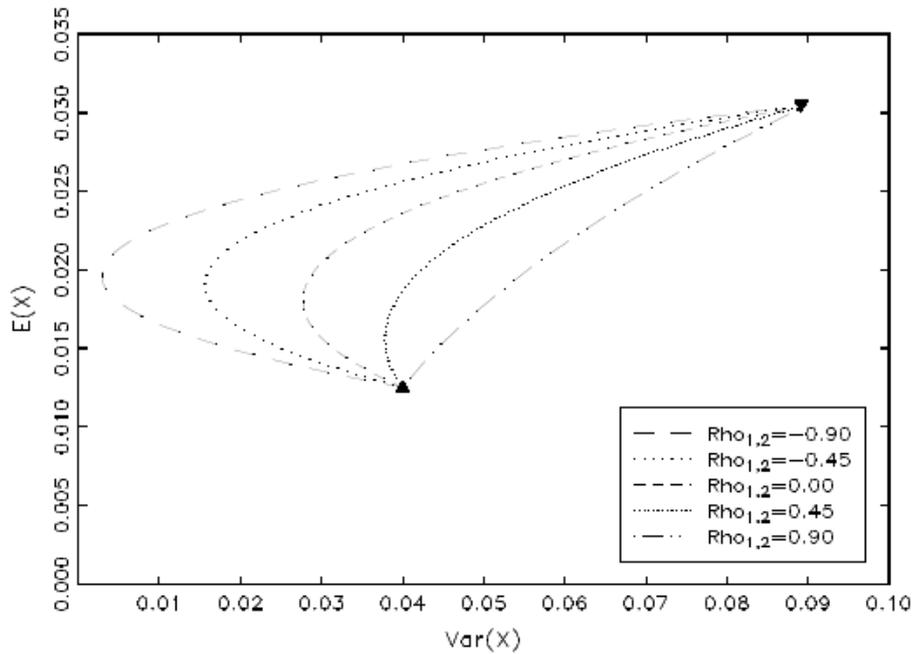
As it is possible to see from the picture “2”, the curve on the graph is the efficient frontier. Starting from the bottom of the curve, the efficient frontier is formed by the portfolios characterized by low risk and by low return. Little by little, the curve gets formed by the portfolios that have higher return and higher risk. All the portfolios below the efficient curve are considered inefficient. What does this mean? It means that when the investor selects a portfolio that is on the efficient frontier, he is choosing the portfolio that for a certain level of return is offering the lowest possible value of risk. Meanwhile a portfolio that is below the hyperbola, for the same rate of return is offering a higher variance. The same happens when the investor is looking for a certain level of risk (standard deviation); there is no portfolio below the efficient frontier that is able to offer a higher return than the one offered on the curve for the same level of risk. If the investor chooses to invest in a portfolio that is not on the efficient frontier, this means that he is investing in a portfolio that is not efficient. All this, highlights the “*mean-variance dominance criterion*” of the model: for any portfolio below the efficient curve, there is one that “dominates” it on the efficient curve (in expected return or risk terms).

Does the efficient frontier tell us which one is the best portfolio for the investor? Absolutely no! The efficient frontier specifies all the possible efficient portfolios that the investor can select. The *optimal portfolio* of the investor will be the one that matches his preferences and, in particular, that maximizes his expected utility function. The utility function expresses the preferences of the investor and its risk aversion. For risk aversion it is meant the risk the investor he is ready to take in order to achieve a certain level of return. So, what will be the portfolio the investor is going to select? In the majority of the cases the solution of the problem is given by the tangency point given by the interception of his utility function and the efficient frontier; that will be the optimal portfolio of the investor.



**Figure 3:** Tangency between the expected utility function and the efficient frontier

In the portfolio optimization, a great role is played by the correlation among the assets, which is one of the main aspects of the diversification. As an example, consider the case in which the investor portfolio is constituted by two risky assets. In this case the efficient frontier will have different slopes, depending on the value of the correlation between the two assets returns. In the following picture it is possible to see the effect of the correlation on the efficient frontier.



**Figure 4:** The role of the correlation

As it is possible to see from the figure, when the correlation is negative the efficient frontier is able to generate a better performance respect the case in which the correlation is positive. In fact the related efficient frontier is more upwards than the latter one. This mean that the utility function of the investor will meet the efficient frontier in tangency point characterized by a higher return given the same level of risk.

### 1.3.1 Description of the optimal portfolio

The optimal portfolio generated by the interception of the expected utility function of the investor and the efficient frontier will have a certain return and risk, which can be described by the following formulas:

- Rate of return:

$$R_P = x_1R_1 + \dots + x_NR_N = \sum_{i=1}^N x_iR_i;$$

where: - “ $x_i$ ” is the weight invested in the “ $i$ -th” asset (the sum must be=1) and “ $R_i$ ” is a random variable representing the return.

- Expected rate of return :

$$\begin{aligned}
 \mathbb{E}(R_P) &= \mathbb{E}(x_1 R_1 + \dots + x_N R_N) = \\
 &= \mathbb{E}(x_1 R_1) + \dots + \mathbb{E}(x_N R_N) = \\
 &= x_1 \mathbb{E}(R_1) + \dots + x_N \mathbb{E}(R_N) = \\
 &= x_1 r_1 + \dots + x_N r_N = \\
 &= \sum_{i=1}^N x_i r_i =: r_P;
 \end{aligned}$$

Note: “ $r_p$ ” can be written in the vectorial notation “ $r_p = x' r$ ”

- The variance :

$$\begin{aligned}
 \text{Var}(R_P) &= \text{Var}(x_1 R_1 + \dots + x_N R_N) = \\
 &= \sum_{i=1}^N x_i^2 \sigma_i^2 + 2 \sum_{i=1}^N \sum_{j=i+1}^N x_i x_j \sigma_{i,j} =: \sigma_P^2.
 \end{aligned}$$

where: “ $\sigma_{i,j} = \rho_{i,j} \sigma_i \sigma_j$ ” ,where “ $\rho_{i,j}$ ” stands for correlation coefficient between asset “ $i$ ” and “ $j$ ”.

“ $\text{Var}()$ ”, is not a linear operator, and the vectorial notation of the variation is  $\sigma_P^2 = x' V x$ , where “ $V$ ” is the variance- covariance matrix.

Finally, the problem of the portfolio selection of the investor can be written as:

$$\begin{aligned} \min_{x_1, \dots, x_N} \quad & \sigma_P^2 \\ \text{s.t.} \quad & \begin{cases} r_P = \pi \\ \mathbf{x}'\mathbf{e} = 1 \end{cases} \end{aligned}$$

which in vectorial notation is:

$$\begin{aligned} \min_{x_1, \dots, x_N} \quad & \mathbf{x}'\mathbf{V}\mathbf{x} \\ \text{s.t.} \quad & \begin{cases} \mathbf{x}'\mathbf{r} = \pi \\ \mathbf{x}'\mathbf{e} = 1 \end{cases} \end{aligned}$$

As it can be seen, the investor looks for the portfolio with the lowest possible value of risk for the expected return he wishes to realize ( $\pi$ ). Markowitz presented his model to the world with these words: “The solution to the model is not only a computing procedure. It is a body of propositions and formulas concerning the shapes and properties of mean-variance efficient sets. I believe that these propositions and formulas have implications for financial theory and practice beyond those of the already widely known special cases”<sup>1</sup>.

His work changed the way the people invested, and at the same the model became of inspiration for many other portfolio strategies. In part they are implementations of the optimization model, in part they are completely new approaches.

#### 1.4 Limits of the Mean- Variance model

Markowitz’s model opened a new era for the portfolio strategy; but the mean-variance model is not exempt to some critiques. The most aspects highlighted by the literature are:

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<sup>1</sup> Harry Markowitz, *Mean Variance Analysis in Portfolio Choice and Capital Markets*, Basil Blackwell, 1987, pp. “Preface”.

⇒ *The estimation error*: The model works with the moments of the asset; the expected return, the variance and the covariance matrix. In this way the model is generally subject to the estimates of the past values of these moments. But this approach is known for producing asset weights that considerably fluctuate over time and they do not perform so good. In fact the estimates are computed using the historical data of the assets; this automatically means that the estimates are linked to the past. In addition to this, a natural question arises: what should be the right length of time to calculate the estimates? So, the parameters that are computed bring with them some errors, and this produces the effect of missallocation: the weights invested in the different assets are not correct. If we knew the “true” parameters (and so we should know the future), the model would work perfectly! But in the real world we never know this information. We just have got estimates to face the future. Some authors argued these limits; Stein (1995)<sup>2</sup> showed that the simple sample statistics is not properly adequate in resolving this kind of problem; Ziemba and Chopra (1993)<sup>3</sup> focus their attention on the fact that the errors in the estimates return are more relevant than variance estimates errors. Finally Michaud (1989)<sup>4</sup>, noted on empirical experiments that the mean-variance optimization overrates the assets with large estimated returns, negative correlation and small variance.

⇒ *The portfolio is static*: the model supposes a structure of this kind: at the beginning of the period the investor invests his capital among the different assets; at the end of the time, the portfolio achieves the return of the strategy. In all this time the investor has

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<sup>2</sup> Bradley Efron and Carl Morris, *Stein's paradox in statistics*, “Scientific American”, (1955), volume 5, pp.119-127.

<sup>3</sup> Viigay K Chopra, William T Ziemba, *The effect of errors in means variances and covariances on optimal portfolio choices*, “Journal of portfolio management”, (1993), pp-6-11.

<sup>4</sup> Richard O. Michaud, *The Markowitz Optimization Enigma: Is Optimized Optimal?*, “Financial Analyst Journal”, January-February (1989), pp.31-42.

to “wait”; he cannot make any intermediate change in the structure of the portfolio. The investor never changes his optimal allocation once he chose it. It does not account for the chance of portfolio rebalancing within the investment horizon. In this way the portfolio is a static portfolio, while investor would need something more dynamic and able to perform along the time. The investor may need a “multi-period” utility model, not focused just on one single period. It should be implemented with a revision strategy of the portfolio and so periodically revisited. Today’s, “Continuous-time models” have been implemented to overcome this problem, through the use of stochastic differential equations and geometric Brownian motion models. But they also present some problems since the parameters are assumed to be insensitive to the movements in the market<sup>5</sup>.

⇒ *Transaction costs*: the model does not take in account the presence of transaction costs and taxation on capital gain. It is true that today the commissions are lower and lower, but however they are still present in the market. From one side this allows the model not to be too much complicated to be applied, but on the other side it does not represent in faithful way the reality. The trading opportunities are not costless. Ignoring transaction costs and the taxation on capital gains can bring to not a right conclusion and to the optimal portfolio.

## 1.5 Naive models

After the “Modern Portfolio Theory” of Harry Markowitz, a new literature is born and it has produced a new large variety of portfolio models. Most of the new models took inspiration from the mean-variance model,

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<sup>5</sup> G. Yin and Xun Yu Zhou, *Markowitz’s Mean-Variance Portfolio Selection with Regime Switching: From Discrete-time Models to Their Continuous-time Limits*, “Insurance: Mathematics and Economics”, (2008), volume 43, pp.456-465.

relying on the assets moments and on the optimization. At the same time, new model arose, taking a different direction and focusing on the “capital allocation”. In other words, these latest models are concentrated on how to allocate the fraction of capital to each asset without taking account of their moments. Then, which is the best strategy? This question has no answer; every model has got its pros and cons, and it is up to the investor to decide which model is the one that better represents his strategy. Today, the literature is still discussing and providing new models to find a strategy that is better than the others, and it is continuously renewing itself. The strategy applied by each model plays a key role in the investor success. At this point, given the modern literature, it has been considered opportune to select and present two naive and simpler models of the capital and the optimal allocation, which have found considerable success and application among the investor choices.

- *Minimum variance model*: this model is a derivation of the Mean Variance strategy, and in some sense it overcomes one of its major problems. This strategy in fact, simply works with the covariance matrix of the assets estimated from the historical data; the portfolio selected by the investor is the portfolio with the minimum variance between all the available portfolios. It completely ignores the estimates of the assets' expected returns. In this way the strategy is not dependent on this feature, and it is not affected by the estimation errors on expected returns. One of the criticisms moved could be that this strategy is a low volatility investment; somebody could think in his mind: “no risk no profit”. But the literature has shown that low volatility portfolios/stocks do not perform worse than high volatility stocks/portfolios (Black, Jensen Scholes in 1972<sup>6</sup>, Haugen and Heins in 1975<sup>7</sup>), most of the times because high return stocks are overrated and so they do not give a

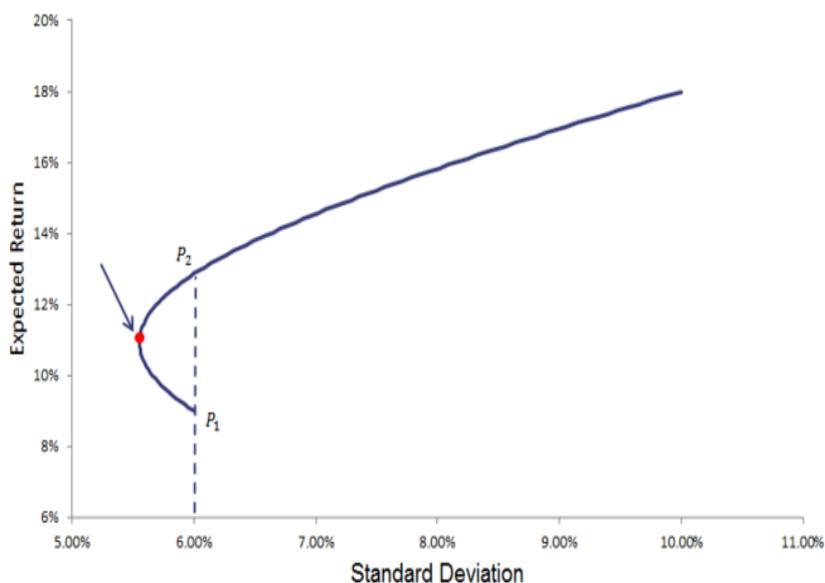
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<sup>6</sup> Fischer Black, Michael C. Jensen and Myron Scholes, *The Capital Asset Pricing Model: Some Empirical Tests*, Praeger Publishers Inc., (1972).

<sup>7</sup> Robert A. Haugen and James Heins, *Risk And The Rate Of Return Of The Financial Assets: Some Old Wine In New Bottles*, “Journal Of Financial And Quantitative Analysis”, (1975), pp.775-784.

good risk-adjusted return; this means that for the level of risk supported by the investor, the return of the stock is not rewardable as one would expect for the stocks with high volatility, while the stocks with low volatility has been the opposite. The portfolio chosen by the minimum variance is the one with less risk on the efficient frontier. Usually this portfolio is often considered, especially as benchmark. The model works with a “moment constriction”; the portfolio chosen is the one that minimizes the variance of return of the portfolio. As said by Guillame Coqueret<sup>8</sup>, “in a post- crisis era portfolios which bear lower risk are likely to be sought by investors”. The drawback of this model is that it generates large negative weights; this means that the investor is in highly positions of short-selling for some securities. However this is not an obstacle for the use of this portfolio, very used as benchmark. The problem to solve to detect the minimum variance portfolio can be expressed as:

$$\begin{aligned} \min_{x_1, \dots, x_N} \quad & x'Vx \\ & x'e = 1 \end{aligned}$$



**Figure 6:** The Minimum Variance portfolio

<sup>8</sup> Guillame Coqueret, *Diversified minimum variance portfolios*, “Annals of finance”, (2014) May, pag. 221-241.

As it is possible to see from the picture, after having built the efficient frontier through the Mean Variance criterion, is selected the Minimum Variance portfolio on the vertex of the hyperbole, that is the efficient portfolio with the lowest possible value of risk among all the possible portfolios.

- *1/N*: This model is quite different from the Mean Variance model, because it based on the allocation of the capital and no more on the assets moments. In this way the portfolio is not affected by the estimation error and the strategy to follow is quite easy to understand and to execute. It does not need any articulated computation and the capital allocation is the result of a simple and heuristic operation. In fact in the *1/N*, known also as “naive strategy” or equally weighted portfolio”, the proportion of capital invested in each asset is the same. So, one of the major advantages of this model is the fact that it is easy to apply even when the number of the assets is very high, while optimization models need more parameters and result more complicated from this point. In addition to this, according to some authors, investors use this simple strategy when they allocate their wealth across securities<sup>9</sup>.The letter “N” is assumed to be the number of assets considered in the investment. The sum of the weights invested in each security must be equal to 1. The weight invested in each asset is equal to:

$$x_i = \frac{1}{N}, \text{ with } i = 1, \dots, N$$

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<sup>9</sup> Victor De Miguel, Lorenzo Garlappi and Roman Uppal, *Optimal versus naive diversification: how inefficient is the 1/N portfolio strategy*, “The Review of financial studies”, (2009) n5 v 22, pp.1915-1953.

The variance of the portfolio is:

$$\sigma_P^2 = \frac{1}{N} \sum_{i=1}^N \frac{\sigma_i^2}{N} + \frac{N-1}{N} \sum_{i=1}^N \sum_{j=1, j \neq i}^N \frac{\sigma_{i,j}}{N(N-1)}$$

In this equation, it is possible to note that first addendum is the mean of the variances of the rates of returns of the financial assets, and that the second addendum represents the mean of the assets covariance between the rates of the financial assets. So we can rewrite the variance of the portfolio as:

$$\sigma_P^2 = \frac{1}{N} \bar{\sigma}_i^2(N) + \frac{N-1}{N} \bar{\sigma}_{i,j}(N)$$

When is considered a multi periodical strategy (as it was defined previously), where in each period the portfolio weights are revised and rebalanced according to the strategy applied by the model, the turnover is the measure that tells to the investor how much “trading” is needed in order to perform that portfolio strategy along the time horizon of the investment considered. If the turnover of the strategy is very high, this means that lot of assets trade is required, and that the transaction costs will have an important impact on the cost of the strategy. Vice versa, if the turnover is low, the transactions cost will have a low impact on the strategy applied.

In the case the 1/N model is applied to an investment where the weights are rebalanced every period (chosen by the investor), the turnover of this strategy is typically very low and it is close to zero. However, the rebalancing portfolio has got some turnover. The aim of the strategy is to keep the weights constant and equal in each

period; in this way, every time the assets hold by the investor increase or decrease in their value, the strategy periodically sells or buys the assets in order to keep the weights fixed. To adjust its weights, on rebalancing dates it sells high and buys lows.

### 1.5 Other literature models

Here is a brief presentation of other widely spread models produced by the literature that found real application the investors investment choices.

- *The value weighted model*

In order to describe the value weighted portfolio-model, it has considered useful to recall the CAPM, the model of general equilibrium of the markets. It is one of the most used in the financial world, and it is built on some very simple assumptions:

- Absence of transaction costs;
- investors are mean-variance optimizers;
- perfect information;
- unlimited risk free borrowing and lending
- infinitely divisible assets;
- investors are price takers and they have homogeneous beliefs.

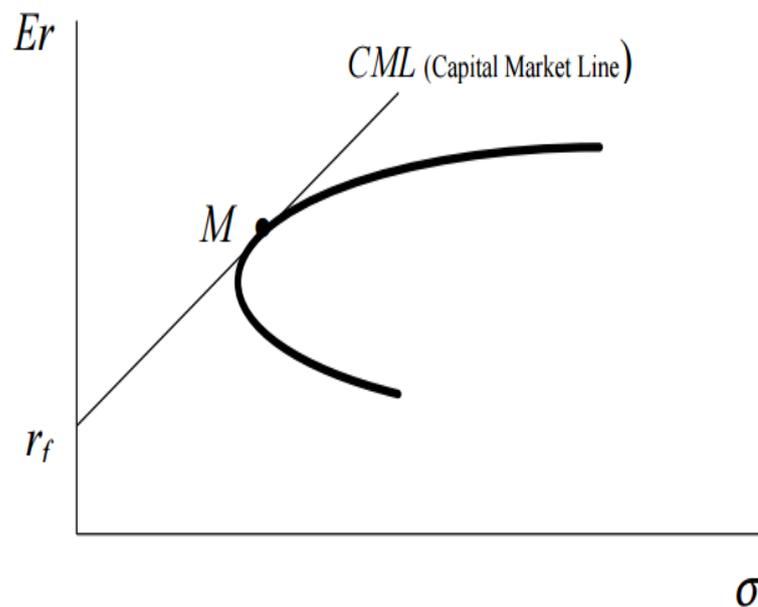
Thanks to this model, it is derived the following formula that explains the return for each asset:

$$R_i = R_f + \beta_i(R_m - R_f)$$

$$\beta_{iM} = \frac{\text{cov}(R_i, R_M)}{\sigma^2(R_M)}$$

where “ $R_f$ ” is the considered risk free, “ $R_m - R_f$ ” is the risk premium (the difference between investing in the market and in the risk

free), and “ $\beta$ ” gives a measure of the asset risk respect the market. The model establishes a direct relation between the return produced by the asset and its risk. Consequently, a high beta stock will be interpreted as a high return for the stock (but also a higher risk). In addition, the CAPM tell us if a financial asset is overrated or underrated (it should be compared the theoretical value with the asset real value). The portfolio hold by the investors is the same for everybody and it is represented in “Figure7”, and is defined as the “market portfolio”. The CAPM has got many applications in financial economy, as for example the calculation of the cost of debt of the firms. The weakest point of the model is its assumptions; they are quite from the real world.



**Figure 7:** Graphical representation of CAPM model.

The value weighted portfolio is built on the environment hypothesis of the CAPM world (it is the market portfolio). It is made of all the assets available and the weights are proportional to the market capitalization of the security (for this it is called “value weighted”). Every time the stock’s price changes, this has an impact on the portfolio value. The *i-th* asset weight of the portfolio is given by the formula:

$$w_{i,t} = \frac{MV_{i,t}}{\sum_{i=1}^N MV_{i,t}}$$

where “ $MV_{i,t}$ ” is the market value of the “*i*” asset at time “*t*”, and “ $w_{i,t}$ ” is the weight of the “*i*” asset.

Most of world indexes, such as the S&P500, FTSE100, NASDAQ are value weighted indexes. According to some researches, this portfolio model is gaining popularity especially in the developed country, while it is less considered in emerging countries.<sup>10</sup> However, many empirical tests have shown that this model is not always the best solution for the investor investment.

- *The 60/40 model*

This portfolio is made of 60%stocks and 40%of bonds. This strategy has long formed the backbone of the traditional asset allocation strategy before the 90’s. One of the criticisms about this model is its weak diversification; in fact it is not well diversified when studied from the prospective of how each asset class contributes to the total portfolio risk. Stocks are usually more volatile than bonds, so their fluctuation has a great impact in the portfolio risk. However in recent years, given the fact that bonds do not provide a

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<sup>10</sup> Uptal Bhattacharya Neal Galpin , *The Global Rise of the Value-Weighted Portfolio*, “ Journal of Financial and Quantitative Analysis”, (2011) , volume 46, pp 737-756.

high return such in the past, is contributing to the decline of the strategy. In this sense the 60/40 model is considered as a “down” strategy, because its use is lower, but not for this reason is considered as “out” from the investors choices.

- *Risk parity model*

This model aims to diversify the investment focusing on the securities’ risks; it means to take the similar quantity of risk in stocks and bonds. This model can be seen as alternative to the 60/40 model which is more focused on a balanced investment rather than looking at a certain level of same risk for the securities selection. In the risk parity model the allocation and the diversification are done by risk and not by dollars. Concretely, this means that the investment is addressed to low-risk assets than in high risk assets. In the 60/40 model instead most of the risk comes from the equity; in fact stocks are usually higher return and are more volatile than bonds, so movement in the stock market dominates the risk in the portfolio. A “risk-parity investor” would say:

*“We do not believe expected returns on equities are high enough to give them a disproportionate part of our risk budget”<sup>11</sup>*

The final goal of the strategy is to earn a certain amount of money with less volatility and risk. It tries to equalize risk by the allocation in a large range of categories (stocks, government bonds, real estates, commodities). In this way the investor can be prepared for different scenarios. Because this strategy involves investing in assets with low risk, the return may be no so high. So, the model can outperform the 60/40 especially in the moment in which the investor applies some leverage (defined as the possibility for the

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<sup>11</sup> Clifford S. Asness, Andrea Frazzini, and Lasse H. Pedersen, *Leverage aversion and risk parity*, “Financial Analyst Journal”, (2012) n1 v68 ,pp. 47-59.

investor to invest more money than his possibility through the use of debt or financial investments) to invest to this strategy. Risk parity strategies are gaining trust among the investors who are rethinking of their traditional allocation strategies.

### **1.6 Is the 1/N strategy really applied by the investors?**

According to the research of Shlomo Bertanzi and Richard H. Thaler, investors follow the diversification principle in their investments and in part they adopt the equally weighted strategy for their investments; they came up to this result analyzing the market of “contribution saving plans”. Contribution saving plans are defined as investment retirement programs in which the investors contributes money on a regular basis with the aim to receive in the future the same amount of money plus some benefits linked at investment chosen by the investor for that money. So in this sense the benefits are not fixed but they fluctuate. The retirement’s programs are an increasing and widespread phenomenon, and investors are given the chance and the responsibility to decide how to allocate their savings among the different opportunities offered to them. In doing this, investors are called to use an investment strategy. This has created some concerns, because people usually lack of financial education in advanced decisions. They are not always financial investors. As a prove of this, a study of John Hancock in financial services in 1995 discovered that the majority of the people were convinced that money market had more risks than government bonds and they thought that their company was safer than a diversified portfolio. In connection with this work, it has been studied which strategy investors follow while deciding their contribution saving plan, paying attention if their choices can be attributed at some portfolio-model. In their research<sup>12</sup>, Shlomo Bertanzi and Richard H.Thaler looked for evidence that investors apply

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<sup>12</sup> Shlomo Bertanzi and Richard H.Thaler, Naive diversification Strategies in Defined Contribution Saving Plans, “*American Economic Association*”, 2001, Vol 91(1), pp.77-98.

strategies based on diversification, even without knowing it. The authors came out also with an expected result. In fact one of the main aims of the survey was to see how many people adopted the heuristic 1/N strategy. To their surprise more than one third of the people interviewed applied the 1/N model. The methodology conducted in the experiment included three tests, differing between them for the approach and their content. The sample interviewed were employees of the California University. In the first test, it was sent an e-mail asking to the “investors” how they would allocate their saving in a contributive saving plan in these three different situations, explaining verbally all the investments strategies.

<i>Situation 1</i>	<i>Situation 2</i>	<i>Situation 3</i>
Stock fund vs Bond fund	Stock fund vs Balanced fund	Balanced fund vs Bond fund

**Figure 8:** Investment opportunities in the first test.  
**Source:** Look at [12].

The opportunities were: a stock fund (investing in big and small companies), a bond fund (investing in high-medium fixed income of government and company bonds) and a balanced fund (half of the assets were invested in stocks, half in bonds). The results showed that in each situation, there was a large part of investors that used the 50-50 strategy (34% in situation A, 21 % in B). In situation C, 28% selected the naive strategy, while the 33% of the interviewed used the balanced fund. Stocks also seemed to be higher demand than Bonds (54% in situation A, 73% in B).

In the next test, conducted through e-mail and showing graphically the historical returns of the 3 different investments, it was added a fourth condition to the interviewed sample:

**Situation 4: Choose among one of these fund**

100%stocks ,75% stocks 25% bonds, 50%stocks 50%bonds , 25%stocks 75%bonds , 100% bonds

**Figure 9:** Investment opportunities in the second test.

**Source:** Look at [12].

Also in this experiment, a large part of investors adopted the 1/N strategy and about the fourth situation most of investors (51%) select the 100% stock allocation, the total percentage of stock allocation equal to 75%. The authors believe that is due to the fact that when investors have to choose one single fund there is no opportunity for the heuristic diversification. Also in this case, stocks seemed to be more requested than stocks.

In the final experiment, conducted verbally, it was asked to the employees to select one fund in two situations. In the first situation there were five funds; four were fixed-incomes (like bonds) and the last was a stock fund. In the other situation there were 4 stocks fund and one fixed income fund. Results; in the first situation the stock fund was selected with a percentage of 43%. In the second condition, stocks funds were selected with a percentage of 68%. The 25% of difference, with a *p-value* equal to 0.01 suggests that the range of funds offered affects the participants. This is an evidence of what was seen in the previous tests. The investors who adopt a naive strategy are substantially influenced by what is offered to them.

According to the authors, the results show that a large part of the investors adopt an equally weighted strategy, establishing the 1/N strategy a model that can be chosen in order to diversify the investments and a financial model really used in the investors' choices.

# CHAPTER 2

## 1/N OR OPTIMIZATION MODELS?

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### 2.1 An ancient rivalry

In the course of the last century, many financial models have been created, implementing the previous ones and trying to overcome their limits. Today there is still no answer about this debate and the literature is continuing to renew itself proposing new models and through the implementation of the traditional ones.

About the 1/N model, it can be thought and interpreted as a very ancient strategy, which precedes of almost 1550 years the work of Harry Markowitz. In fact already in Jewish culture it is possible to find the principle of equally weighted strategy. The “*Talmud*” is an ancient holy Jewish book which contains all the traditions, culture and laws of this population. In this book, there is an ancient quote about the 4<sup>th</sup> century that declares: “*One should always divide his wealth into 3 parts: a 1/3 in the land, a 1/3 in trades and a 1/3 in his hand*”. This confirms to us that the naive strategy is not something new, but already in the past people were considering and applying it. Maybe is not a very sophisticated and complex strategy, but is a strategy that today has many supporters (also the same Markowitz!). Almost 1600 years have passed and still it is a widely spread and considered strategy.

This very naive technique, sometimes defined also as rough and impulsive, has started a general “rivalry” with the optimization models, which are more sophisticated and some important studies behind them. As it will be said at the end of this chapter, the same Markowitz, who is the godfather of the optimization models, has followed a naive strategy to allocate his savings for psychological reasons.

## **2.2 Literature Research**

There are many articles in the literature that try to demonstrate the superiority of a certain model respect the others, supporting the argumentation with some empirical experiments. We have selected 2 articles that in particular analyze the performance of the 1/N model and the optimization models, arriving at two different conclusions.

### **2.2.1 The outperformance of the naive portfolio**

In the 2009, three authors arrived to the conclusion of the outperformance of the naive model respect the optimization models in their research. In the article: “Optimal versus naive diversification: how inefficient is the 1/N portfolio strategy?”, Victor DeMiguel, Lorenzo Garlappi, Roman Uppal conducted an empirical test to evaluate the performance of 14 models against the naive strategy (1/N) among. The analyzed models belonged to the optimization models family; starting from the Mean-Variance model of Harry Markowitz to some of its implementations, designed to reduce the problem of the estimation error. The major criticism moved and highlighted in the article is in fact the notorious weak point of the optimization models: the estimation error, produced by estimating the parameters from the historical assets sample, causing the effect of fluctuating weights of the portfolio created and the poor performance of it. The equally weighted strategy is used as benchmark in the evaluation of these models for 2 reasons:

- It is easy to use because it does not need the moments estimation and for this reason it cannot be affected by the consequent error typical of the estimation;
- although the sophisticated models born in the last 50 years, investors continue to use simple rules to allocate their money (Bertanzi and Thaler, 2001).

*The models analyzed:* in the following picture are presented the models originally compared by the authors, accompanied by their abbreviation (useful for the comparison of the results). We will not enter in the technical details of the models used, and the interested lector can find more details in the mentioned article.

<b>STRATEGIES</b>	<b>ABBREVIATION</b>
<b>Naive:</b> -1/N	1/N
<b>Classical approaches:</b> -Mean-Variance -In sample Mean Variance	mv mv (in sample)
<b>Bayesian approach to estimation error:</b> -Bayes-Stein -Bayesian Data and Model	bs dm
<b>Moment restriction:</b> -Minimum Variance -Value –weighted market portfolio -MacKinlay and Pastor's missing factor model	min vw mp
<b>Portfolio constraints:</b> -Sample Mean Variance with shortsale constraints -Bayes-Stein with shortsale constraints -Minimum Variance with shortsale constraint -Minimum variance with generalized constraints	mv-c bs-c min-c g-min-c
<b>Optimal combination of portfolios:</b> -Kan and Zhou's "three fund" model -Mixture of Minimum Variance and 1/N	mv-min ew-min

**Figure 10:** List of the models analyzed  
**Source:** Look at [9].

Among the different models, it is important to mention the “*In sample Mean Variance model*”, indicated as “*mv (in sample)*”; this model is built on the “true parameters” of the assets. In particular, while for the other models the assets weights are computed using the parameters obtained from the historical data of the assets and according to them are periodically rebalanced, this particular model uses the parameters computed at the end of each period; it is made to see how the portfolio

would perform if it knew the real parameters, and so knowing the market movements of that period of time considered. Of course, it is impossible to build a portfolio of this kind. But is useful to see how the optimal portfolio would perform knowing exactly the true parameters.

*The data-set:* in the following table are presented the databases used in the experiment, providing also information about the considered time period and the abbreviation used; the interested lector can find more details in the mentioned article. Every dataset included monthly excess returns over the 3 month American Treasury bill.

DATABASE	TIME PERIOD	ABBREVIATION
Ten sector portfolios of the S&P 500 and the US equity market portfolio	01/1981-12/2002	S&P Sectors
Ten industry portfolios and the US equity market	07/1963-11/2004	Industry
Eight Country indexes and the world index	01/1970-07/2001	International
SMB,HML portfolios and the US equity market portfolio	07/1963-11/2004	MKT/SMB/HML
Twenty size-and book to market portfolio and the US equity MKT	07/1963-11/2004	FF-1 factor
Twenty size-and book to market portfolio and MKT, SMB, HML portfolios	07/1963-11/2004	FF-3 factor
Twenty size-and book to market portfolio and MKT, SMB, HML and UMD portfolios	07/1963-11/2004	FF-4 factor

**Figure 11:** List of the databases used  
**Source:** Look at [9].

*The measures of performances:* to evaluate the performance of each strategy among the different datasets, the authors have used the following indexes to compare the relative portfolios created by them; (i) the Sharpe ratio index, (ii) the certainly equivalent return and (iii) the portfolio turnover.

- (i) Sharpe ratio test: it is one of the most widely used indexes when comparing different portfolios and it can help the investor to have an idea of the competitiveness of the portfolio considered. In the experiment, given the time series of monthly out of sample returns generated by each strategy and in each database, the sample Sharpe ratio was computed using the sample mean and the sample standard deviation of the return.

$$\widehat{SR}_k = \frac{\hat{\mu}_k}{\hat{\sigma}_k} ,$$

where “ $\mu_k$ ” is the over excess return under strategy  $k$  in a certain dataset, and “ $\sigma_k$ ” is its standard deviation (the risk free used was a 90 days nominal T-bill).

- (iii) CEQ: certainly equivalent return that considers the coefficient of risk aversion of the investor (“ $\gamma$ ”, in this case equal to 1), defined as the additional margin return the investor needs to accept more risk. The CEQ is equal to:

$$\widehat{CEQ}_k = \hat{\mu}_k - \frac{\gamma}{2} \hat{\sigma}_k^2$$

- (iii) Turnover: the turnover gives to the investor an idea of how frequently the securities are sold and bought in order to follow the strategy along the time horizon of the investment.

$$\text{Turnover} = \frac{1}{T - M} \sum_{t=1}^{T-M} \sum_{j=1}^N \left( |\hat{w}_{k,j,t+1} - \hat{w}_{k,j,t}| \right) ,$$

where “ $w_{k,j,t+1}$ ” is the weight of asset “ $j$ ” at time “ $t$ ” under the strategy “ $k$ ”, “ $w_{k,j,t}$ ” is the portfolio weight before rebalancing in asset “ $j$ ” before the rebalancing at time “ $t$ ”, while “ $w_{k,j,t+1}$ ” is the portfolio weight after the adjustment. “ $T-M$ ” is the difference

between the number of observations (for considered “*i*” asset) and the estimation window used (it will be presented successively).

*Time strategy and rolling window:* the portfolio created by each strategy was hold for the time range of each dataset. In order to select the optimal weights for the creation of the relative portfolio, two rolling windows were used (indicated with letter “*M*”); one based on the past 60 months and the other on the past 120 months. Every month the portfolio was rebalanced using the parameters estimated in the *last previous 60-120 months*. For example, in the 61<sup>th</sup> months were used the estimates of the 60 previous month. In the 62<sup>th</sup> month, were used the parameters estimated along the 60 previous months (from month 2 to month 61). This operation was repeated till all the assets return series was covered (indicated with letter “*T*”).

*Description of the model used:* almost all the models considered in the experiment delivered portfolio weights that can be expressed in the following equation:

$$w_t = \frac{\Sigma_t^{-1} \mu_t}{\mathbf{1}_N \Sigma_t^{-1} \mu_t} ,$$

where: : “ $\mu_t$ ” is used to denote the expected returns on the risky asset in excess of the risk free asset; “ $\Sigma_t$ ” is the variance-covariance matrix of returns;  $\mathbf{1}_N$  defines an N-dimensional vector of ones and “ $w_t$ ” is the vector of portfolio weights invested in the N risky assets.

*The results:* are now reported the different strategies results evaluated under the three different indexes Sharpe ratio, CEQ and turnover among the considered datasets. The authors, decided to report the results just for the estimation window of 120 months (in fact the results for the other estimation window were not very different).

**Sharpe ratios for empirical data**

Strategy	S&P sectors $N = 11$	Industry portfolios $N = 11$	Inter'l portfolios $N = 9$	Mkt/ SMB/HML $N = 3$	FF 1-factor $N = 21$	FF 4-factor $N = 24$
1/N	0.1876	0.1353	0.1277	0.2240	0.1623	0.1753
mv (in sample)	0.3848	0.2124	0.2090	0.2851	0.5098	0.5364
mv	0.0794 (0.12)	0.0679 (0.17)	-0.0332 (0.03)	0.2186 (0.46)	0.0128 (0.02)	0.1841 (0.45)
bs	0.0811 (0.09)	0.0719 (0.19)	-0.0297 (0.03)	0.2536 (0.25)	0.0138 (0.02)	0.1791 (0.48)
dm ( $\sigma_a = 1.0\%$ )	0.1410 (0.08)	0.0581 (0.14)	0.0707 (0.08)	0.0016 (0.00)	0.0004 (0.01)	0.2355 (0.17)
min	0.0820 (0.05)	0.1554 (0.30)	0.1490 (0.21)	0.2493 (0.23)	0.2778 (0.01)	-0.0183 (0.01)
vw	0.1444 (0.09)	0.1138 (0.01)	0.1239 (0.43)	0.1138 (0.00)	0.1138 (0.01)	0.1138 (0.00)
mp	0.1863 (0.44)	0.0533 (0.04)	0.0984 (0.15)	-0.0002 (0.00)	0.1238 (0.08)	0.1230 (0.03)
mv-c	0.0892 (0.09)	0.0678 (0.03)	0.0848 (0.17)	0.1084 (0.02)	0.1977 (0.02)	0.2024 (0.27)
bs-c	0.1075 (0.14)	0.0819 (0.06)	0.0848 (0.15)	0.1514 (0.09)	0.1955 (0.03)	0.2062 (0.25)
min-c	0.0834 (0.01)	0.1425 (0.41)	0.1501 (0.16)	0.2493 (0.23)	0.1546 (0.35)	0.3580 (0.00)
g-min-c	0.1371 (0.08)	0.1451 (0.31)	0.1429 (0.19)	0.2467 (0.25)	0.1615 (0.47)	0.3028 (0.00)
mv-min	0.0683 (0.05)	0.0772 (0.21)	-0.0353 (0.01)	0.2546 (0.22)	-0.0079 (0.01)	0.1757 (0.50)
ew-min	0.1208 (0.07)	0.1576 (0.21)	0.1407 (0.18)	0.2503 (0.17)	0.2608 (0.00)	-0.0161 (0.01)

**Figure 12:** Sharpe ratio results  
**Source:** Look at [9].

In the first column are represented all the strategies with the correspondent abbreviation; in the following columns are reported the Sharpe ratio results for the correspondent database (the results for the “FF-3 factor” database were not reported by the authors because they were very similar to the results of the “FF-1” database). For more details look at picture the Evaluating the Sharpe ratio performances, it is evident that the Sharpe ratio of the naive strategy outperforms the ones of all the other considered models, and in the few cases it was “defeated” the difference among the Sharpe values was not statistically significant (in the brackets is represented the *p-value* of the difference between the 1/N and the other strategies). As it is possible to see, the “in sample mv” has achieved a very high performance, due to the fact that it was built with the true parameters. Comparing it especially with the Mean Variance model it is possible to note that there is an important difference of result, because of the estimation errors. For a more general strategies

comparison, it is possible to evaluate the differences on their Sharpe value; this gives an idea of the loss. A good result is achieved by the Minimum Variance model (represented as “*min*”); but its *p-values*, compared with 1/N, are not statistically significant.

**Certainty-equivalent returns for empirical data**

Strategy	S&P sectors <i>N</i> = 11	Industry portfolios <i>N</i> = 11	Inter'l portfolios <i>N</i> = 9	Mkt/ SMB/HML <i>N</i> = 3	FF 1-factor <i>N</i> = 21	FF 4-factor <i>N</i> = 24
1/ <i>N</i>	0.0069	0.0050	0.0046	0.0039	0.0073	0.0072
mv (in sample)	0.0478	0.0106	0.0096	0.0047	0.0300	0.0304
mv	0.0031 (0.28)	-0.7816 (0.00)	-0.1365 (0.00)	0.0045 (0.31)	-2.7142 (0.00)	-0.0829 (0.01)
bs	0.0030 (0.16)	-0.3157 (0.00)	-0.0312 (0.00)	0.0043 (0.32)	-0.6504 (0.00)	-0.0362 (0.06)
dm ( $\sigma_{\alpha} = 1.0\%$ )	0.0052 (0.11)	-0.0319 (0.01)	0.0021 (0.08)	-0.0084 (0.04)	-0.0296 (0.00)	0.0110 (0.11)
min	0.0024 (0.03)	0.0052 (0.45)	0.0054 (0.23)	0.0039 (0.45)	0.0100 (0.12)	-0.0002 (0.00)
vw	0.0053 (0.12)	0.0042 (0.04)	0.0044 (0.39)	0.0042 (0.44)	0.0042 (0.00)	0.0042 (0.00)
mp	0.0073 (0.19)	0.0014 (0.05)	0.0034 (0.17)	-0.0026 (0.04)	0.0054 (0.09)	0.0053 (0.10)
mv-c	0.0040 (0.29)	0.0023 (0.10)	0.0032 (0.29)	0.0030 (0.28)	0.0090 (0.03)	0.0075 (0.42)
bs-c	0.0052 (0.36)	0.0031 (0.15)	0.0031 (0.23)	0.0038 (0.46)	0.0088 (0.05)	0.0074 (0.44)
min-c	0.0024 (0.01)	0.0047 (0.40)	0.0054 (0.21)	0.0039 (0.45)	0.0060 (0.12)	0.0051 (0.17)
g-min-c	0.0044 (0.04)	0.0048 (0.41)	0.0051 (0.28)	0.0038 (0.40)	0.0067 (0.17)	0.0070 (0.45)
mv-min	0.0021 (0.07)	-0.2337 (0.00)	-0.0066 (0.01)	0.0044 (0.28)	-0.0875 (0.00)	-0.0318 (0.07)
ew-min	0.0037 (0.04)	0.0052 (0.42)	0.0050 (0.24)	0.0039 (0.43)	0.0093 (0.12)	-0.0002 (0.00)

**Figure 13:** Certainly equivalent return results.  
**Source:** Look at [9].

Under the analysis of the certainly equivalent returns, the results confirm the outperformance of the equally weighted strategy; the 1/N model is still the model best-performing, and in few cases it was outperformed. But also in these cases the differences were not statistically significant.

**Portfolio turnovers for empirical data**

Strategy	S&P sectors $N = 11$	Industry portfolios $N = 11$	Inter'l portfolios $N = 9$	Mkt/ SMB/HML $N = 3$	FF- 1-factor $N = 21$	FF- 4-factor $N = 24$
$1/N$	0.0305	0.0216	0.0293	0.0237	0.0162	0.0198
Panel A: Relative turnover of each strategy						
mv (in sample)	–	–	–	–	–	–
mv	38.99	606594.36	4475.81	2.83	10466.10	3553.03
bs	22.41	10621.23	1777.22	1.85	11796.47	3417.81
dm ( $\sigma_d = 1.0\%$ )	1.72	21744.35	60.97	76.30	918.40	32.46
min	6.54	21.65	7.30	1.11	45.47	6.83
vw	0	0	0	0	0	0
mp	1.10	11.98	6.29	59.41	2.39	2.07
mv-c	4.53	7.17	7.23	4.12	17.53	13.82
bs-c	3.64	7.22	6.10	3.65	17.32	13.07
min-c	2.47	2.58	2.27	1.11	3.93	1.76
g-min-c	1.30	1.52	1.47	1.09	1.78	1.70
mv-min	19.82	9927.09	760.57	2.61	4292.16	4857.19
ew-min	4.82	15.66	4.24	1.11	34.10	6.80

**Figure 14:** Turnover results.

**Source:** Look at [9].

For the last performance of evaluation, the results show that the equally weighted strategy has got a better turnover respect to the other models. The only exception is the value weighted portfolio (that was described in the first chapter) which has got a turnover equal to zero. Entering in details, among the optimizing models is possible to see that for the Mkt/SMB/HML database the turnover is very low with respect the other models; this because the number of assets is very low ( $N=3$ ) and because two of them are active and supervised portfolios. Comparing the Minimum Variance with the Mean Variance strategy, the former substantially outperforms the latter. In addition to this, is evident that the strategies with short-sale constraints have lower turnover than the other models.

*Conclusions:* Thanks to the results of this empirical test the authors arrived to two conclusions. The first is that the  $1/N$  model outperformed

all the models in Sharpe Ratio, Certainly equivalent return and Turnover terms. This happened especially when the number of assets was very large and when the historical data of the assets were not sufficiently long, so that was not possible to estimate precisely the parameters. In the few cases in which the naive strategy achieved lower values in the different measures of performance, the difference was not statistically significant. Is it possible to find an answer to this result? Yes. All the models analyzed, from the first model (the traditional Mean-Variance) till the 14th one (an optimal combination of portfolios made by Garluppi and Wang), need the estimates of the assets moments to work efficiently. This automatically involves the typical weakness of the optimization models: *the estimation error*, due to the fact that the true moments are unknown and they cannot be computed. In fact in this case, because they are coming from a historical data series of 120 months, the mean and the variance are just estimates and not the true values of the parameters. The impact of the estimation error on the final result is the misallocation effect; the weights are not allocated properly among the different assets. As shown, this problem automatically may lead to a loss in terms of performance. This is the point that De Miguel, Garlappi and Uppal (and also other authors in the literature) highlight in their article; starting from this point, they demonstrated that the naive strategy is better than the other models, because of the lack of the estimation error. As second result, the authors discovered that a large part of the estimation error can be blamed to the estimation of the mean (respect to the covariance estimation), and that the traditional Mean Variance portfolio when constituted of 25 assets, in order to perform better than the 1/N should need an estimation window equal to 3000 months! This means something like 3 centuries of historical data! Of course, at least today this is impossible.

## 2.2.2 The outperformance of the optimization models

Later on, the article received some critics; in fact some authors replied to this result arguing that the empirical test did not rely on reasonable assumptions and so the results were not reliable. In the 2010 in fact it was published: “In defense of optimization: the fallacy of 1/N”<sup>13</sup>, written by Kritzman, Page and Turkington. As the title of the article suggests, the authors called into question the previous research, analyzing deeply the test that was conducted. The test, run over the 14 models, showed some weak points about the criteria it adopted. The argumentation was supported by three key points;

- First of all, the typical estimation error of the optimization models had not to be overrated; in fact, as Kritzman showed in one of his previous researches<sup>14</sup> about the portfolio sensibility, it is not affected by small estimates errors. In fact when dealing with assets that have similar expected return and risk, the errors in the estimate of these values may produce the misallocation effect, but the final return, either in the correct portfolio either in the “wrong one”, will be very similar. The same result happens when the assets have dissimilar expected return and risk. So in this sense the estimation error cannot be exaggerated.
- In his portfolio theory, Markowitz never said that the investor forecasts should be imposed by the past. The future performances are not necessarily the past performances, but the past can be a tool for the future. Through the observation of the past, the investor can form some beliefs that can help him in the choice of the portfolio to select. That is the reason of the importance of the extrapolation of historical values.

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<sup>13</sup> Mark Kritzman, Sebastien Page and David Turkington, *In defense of optimization: the fallacy of 1/N*, “Financial analysts journal”, (2010) March/April, pp.31-39.

<sup>14</sup>Mark Kritzman, *Are optimizers error maximizers?*, “The Journal of portfolio management”, (2006), Summer, pp. 66-69.

- The estimation period used was too much short: no reasonable investor would use parameters estimated over such short samples to invest his money.

Given all the mentioned reasons, the authors decided to make a new test described in the following steps.

*The experiment:* the authors decided to apply a new a test in order to verify if the result achieved by the previous authors was reliable. Especially they conducted a new experiment comparing the 1/N model, the market portfolio and the optimized portfolios based on the inputs needed. For the construction of these optimal models, the expected returns and volatilities were forecasted on the information available in the moment of the portfolio creation. The authors solved for the assets weights that maximized:

$$\sum_{i=1}^n w_i E(r_i) - \lambda \sum_{i=1}^n \sum_{j=1}^n w_i w_j \sigma_{ij}$$

subject to

$$\sum_{i=1}^n w_i = 1 \text{ and } w_i \geq 0, \forall i.$$

The variables that appear have already been defined in the previous experiment, and the only new variable that is “ $\gamma$ ”, the expresses the risk aversion coefficient of the investor defined as the additional margin return the investor needs to accept more risk.

*Dataset:* in the following picture are reported the databases used for the experiment. Datasets were grouped for asset class, beta and alpha; alpha can be seen as the excess rate of return of an asset or portfolio over the return forecasted by a certain model, such as the CAPM. This is the

classification is usually followed by institutional investors. The interested investor can find more information about the databases in the mentioned article.

Dataset	Start Date	End Date
<i>Asset/liability management</i>		
7 asset classes + liability proxy	Feb 1973	Dec 2008
<i>Betas</i>		
10 industries	Jul 1926	Dec 2008
30 industries	Jul 1926	Dec 2008
10 size deciles	Jul 1926	Dec 2008
10 book-to-market deciles	Jul 1926	Dec 2008
10 dividend yield deciles	Jul 1927	Dec 2008
10 momentum deciles	Jan 1927	Dec 2008
10 long-term-reversal deciles	Jan 1931	Dec 2008
10 short-term-reversal deciles	Feb 1926	Dec 2008
<i>Alphas</i>		
500 stocks	Dec 1998	Dec 2008
21 commodities	Jan 1971	Dec 2008
14 hedge fund styles	Jan 1996	Dec 2008
15 asset managers	May 1987	Dec 2008

**Figure 15:** List of the databases used.  
**Source:** Look at [13].

*Time strategy:* in each strategy, the portfolio was subject to a multi-period revision. This means that at the end of each considered period, the portfolio was rebalanced on the basis of the available information. The strategies followed were:

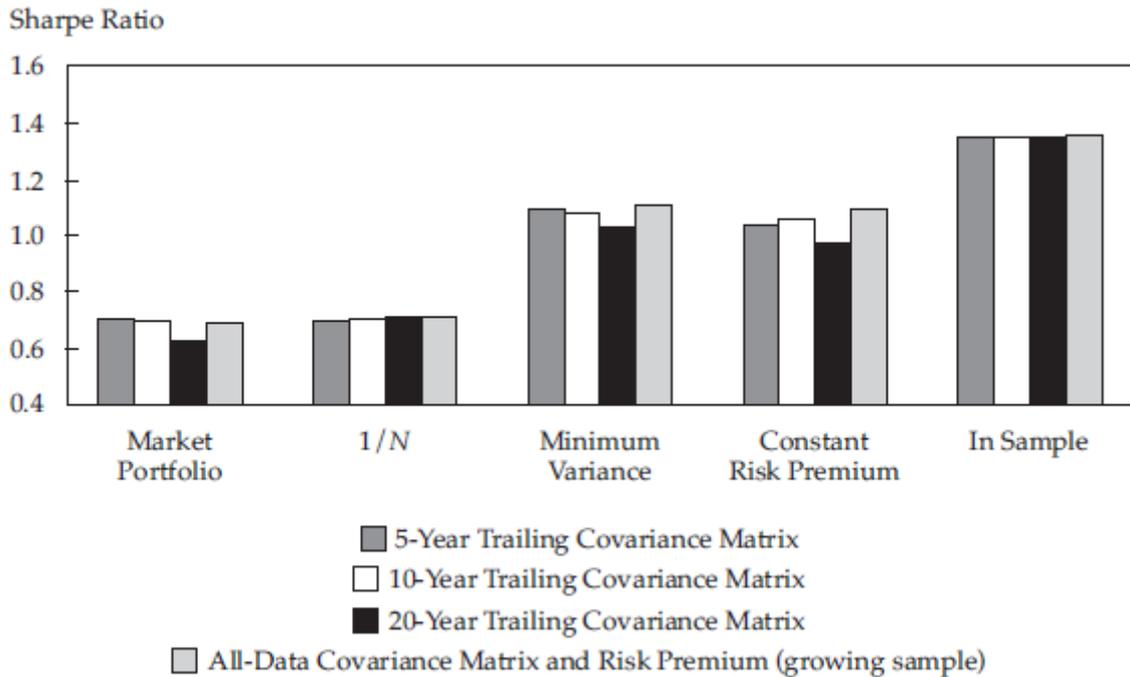
- For the asset/liability management, the portfolio was hold for 5 years and it was rebalanced every year.
- For the other categories of financial instruments, the portfolio was hold for 5 years and it was rebalanced every month.

*Parameters estimation:* differently from the previous experiment, the authors decided to adopt a new strategy for the computation of the parameters, required by the optimization models. About the expected return of the assets, the authors did not rely on the traditional way of calculation from historical data, but from through three different methods:

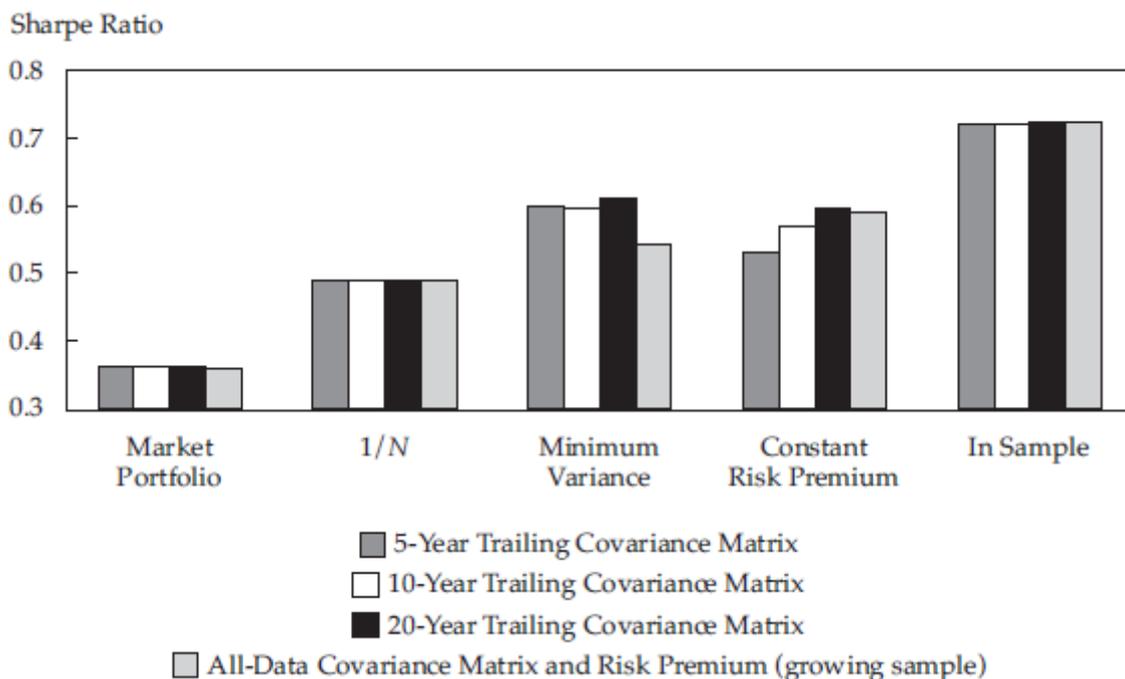
1. In the first method, the returns generated are those of the minimum variance portfolio. In this way the returns generated were obtained on the basis of the extrapolation of the covariance matrix. The returns were kept constant for all securities.
2. In the second method, for every asset was estimated a risk premium based on a consistently historical sample. The return was kept constant for all the duration of the test.
3. Finally, a more statistical approach: the authors used a growing-sample with all the available out of sample data. In fact, when the sample is larger and larger, the estimates obtained from the sample are considered to tend to the true values.

For the estimation of the volatilities it was used a monthly rolling of five, ten and twenty year covariance matrix (except for the “*Alpha category*” where due to the lack of information it was used the 5 year covariance matrix).

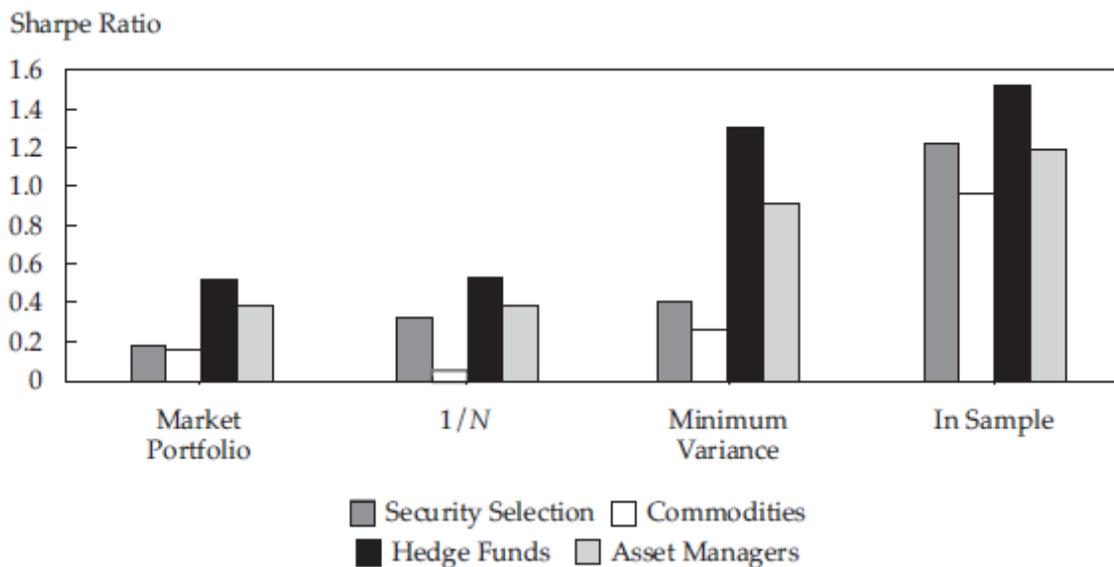
*Results and conclusions:* In the three following tables are reported the experiment results for the three portfolio categories considered by the authors. The results obtained by the authors are very different form the ones obtained by DeMiguel, Garlappi and Uppal. For a more deeper analys, it has also been built the “*In sample*” model, that showed how the optimal would work knowing exatly the true moments estimation of the considered securities.



**Figure 16:** Sharpe ratio results of the “Asset class”  
**Source:** Look at [13].



**Figure 17:** Sharpe ratio results of the “Beta category”  
**Source:** Look at [13].



**Figure 18:** Sharpe ratio results of the “Alpha category”.  
**Source:** Look at [13].

As it is possible to see from the graphs, it is evident the outperformance with respect to the equally weighted portfolio of the optimized portfolios. In particular, in all the three portfolio categories, the Minimum Variance model expressed an interesting performance; in fact in some cases it outperformed the optimization based on the risk premium return. So, working just with the covariance matrix of the assets could be enough to outperform a model that uses more input, as the assets expected returns. According to the authors this can be attributed to the fact that past information is not predictive for the future (especially when considering the expected return). Secondly, although the models used the estimated expected this is not sufficient for portfolio characterized by high return and high risk to achieve a high Sharpe ratio (unless some leverage is applied). Finally, referring to the research of Blitz and Van Vilet (2007)<sup>15</sup>, in an inefficient market the Minimum Variance outperforms the other models because stocks with low risk earn high-risk adjusted return. In fact, stocks with high return and so higher risk usually are bought by

<sup>15</sup> David Blitz and Pim Van Vilet, *The Volatility Effect: Lower Risk Without Lower Return*, “Journal of Portfolio Management”, (2007), fall, pp. 102-113.

manager investors (in order to generate high average return). In this way these assets become overpriced, while stocks with low risks may become underpriced.

In addition to this, the experiment showed that the risk concentration, that is the portfolio volatility divided by the weighted average volatility of the stocks, confirmed the results. On average, it was about 56% for the optimized portfolios and 63% for the 1/N. All the results showed the better performance of the optimization models respect the naive strategy. The authors finally provide an advice for the parameters estimation. According to them the formulas often used to calculate the variance and covariance of the assets from the whole sample, it gives the same importance to market moments with no events as to market moments with significant events. In this way the parameters are indifferently obtained from particular times of the market (like shortfall) and from “normal and noisy” times. To avoid this problem, the sample could be divided into two sub-samples: the first related to normal periods and the second related to turbulent periods; then estimate the assets variance and covariance of these two periods, overweighting the covariance matrix of the market characterized by particular moments. This will improve the portfolio flexibility to significant market events.

### **2.3 Low volatility anomaly: high return with low risk**

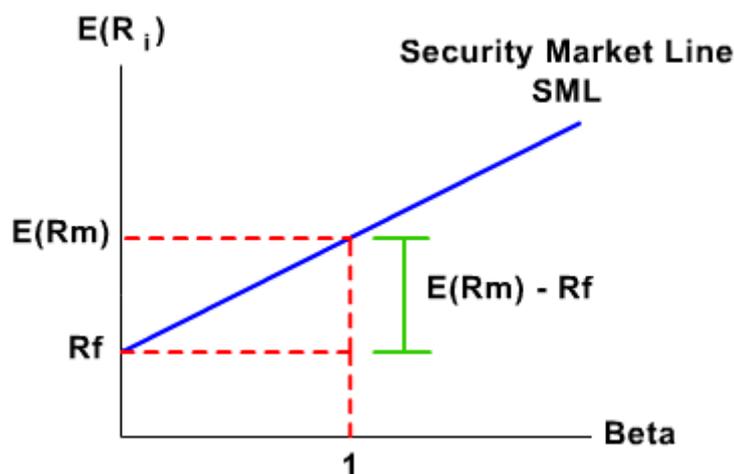
After the big crisis of 2008, the concept of risk has more and more been seen as something with a negative connotation, and investor tries to avoid it investing in safer solutions. For this reason, one of the more requested portfolios is nowadays the “*Minimum Variance*”. This model selects the assets such that the risk supported by the investor is the least among all the possible portfolios on the efficient frontiers. Should an investor who is looking for an investment with a high return to select a portfolio constituted in large part of stocks with low risk and low expected return (and so low beta) instead of stocks with high risk but high

expected returns? One could answer intuitively that the return realized by the portfolio created is very low if also the stocks selected have low risk (“no risk no gain”); this is what the portfolio theory has thought to us till now, because the efficient frontier built is always a growing hyperbola. But in reality this is not always true! The literature about the CAPM model in fact has shown that a portfolio of this kind is able to outperform the other strategies, shown by the empirical works of some authors such as Black (1972), Haugen and Heins (1975).

In their works they discovered that the assumptions of the CAPM lead to some errors in the asset forecast return and that in the market it is present a low volatility anomaly; portfolios with low risk have given higher risk adjusted returns than portfolios with higher stock risks. According to the CAPM model in fact, the return is given by the formula previously described in chapter “1”, and defined as:

$$R_i = R_f + \beta_i(R_m - R_f) .$$

This “equilibrium” introduced by the model, can be graphically be represented in the following picture.



**Figure 19:** The CAPM model graphical representation.

As known, there exists a linear relation between beta and the return, and the intercept on the graph of this model should be equal to the risk free rate. So the slope of the line in the graph is dependent on these two features. According to it, to high beta corresponds to higher return; to low beta corresponds to lower return.

In a study of 1972, Black and other authors showed that for the American market, if an investor could not borrow at a risk free rate, resulting slope of the security market line (SML) is flatter than the one predicted by the CAPM model, so high betas do not deliver so much high returns as was thought. This was shown through some empirical tests.

Later on, in 1975, Haugen and James Heins, gave support to the failure of the CAPM, when they showed through an empirical test conducted over 1926 to 1971, that on a long term investment, portfolios made with safer stocks, produced higher average returns than the portfolios made with riskier assets.

In related study and in support of this theory but with a different sample, Ang, Hodrick, Xing & Zhang (2006)<sup>16</sup> performed a test over the 1960-2000 period using the US stocks, and also their work shows that portfolios based on high volatility earned a very low return.

Other evidences on this topic have been brought to the literature by Baker (2001), Chan, Karceski and Lakonishok (1999), Jangannathan and Ma (2003) and many others.

Finally, Blitz and Vain (2007) showed this low volatility anomaly on a larger sample, considering not just the American region but also the Japanese, the European and for the global market. The result is that the volatility anomaly is persistently present in all the regions for the risk adjusted prospective (for the Europe a Sharpe ratio of 0.49 against the 0.28 for the market, in Japan 0.38 against the 0.18 and for the US 0.58

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<sup>16</sup> Andrew Ang, Robert J.Hodrick, Yuhang Xing and Xiaoyan Zhang, *The Cross-Section of Volatility and Expected Returns*, "The Journal of Finance", (2006), February no 1, pp. 259 -299.

against 0.47). One of the possible reasons of the good performance of the low beta stocks is the fact that assets managers are convinced to buy stocks with high beta (because according to the CAPM they will deliver high returns), but in this way these assets are overpriced respect to low beta stocks. Always reporting the argumentation of Blitz, a further reason could be the fact that, according to the behavioral market portfolio theory developed by Shefrin and Statman (2000)<sup>17</sup>, investors think in terms of a two layers portfolio. There is a low aspiration layer that aims to avoid poverty and losses. In this first act they behave as risk-averse investors. But at the same time they look for a portfolio that can generate some good profits and become risk neutral or also risk-seeking. So investors will overpay for risky-assets, perceived to be the winning solution of their problem. The portfolio will be constituted by many risky stocks and by few low risk stocks (in order to protect their investment). The deviation from risk-averse behavior produces assets with high volatility to be overpriced. So the volatility effect is caused by the errors of private investors.

All these evidences present and observed in the market suggest why strategies with a low risk, like the Minimum Variance model, can be able to generate good and better performances respected an overrated portfolio of high return and risk stocks. Given the recent results, there is a real concern that stocks with low volatility could become more expensive; a possible dynamic that would erode the possible advantages and gains of these strategies.

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<sup>17</sup> Hersh Shefrin and Meir Statman, *Behavioral Portfolio Theory*, "The Journal of Financial and Quantitative Analysis", (2000), June no.2, pp.127-151.

## 2.4 A psychological approach to the matter

Gerd Gigerenzer is a German psychologist specialized in heuristic methods about the process of decision making. In his works, such as *“Simple Heuristics That Make Us Smart”* or *“Bounded Rationality: The Adaptive Toolbox”*, the author has many times spoken about also the financial problems affecting the people, trying to give them an explanation and a solution. The heuristic thought (from the Greek word: εὐρίσκω, heurisko= to find) believes in an approach to the problem through practical methods that maybe are not perfect or optimal, but that provides a solution that is satisfactory and immediate. It does not need many complicated solutions and it requires less effort than other sophisticated methods.

In one of his works, *“Imparare a rischiare”*<sup>18</sup>, Gigerenzer deals with the problem of the optimization. From a psychological aspect, when we face a complex problem our instinct is to find a complex situation to the problem; this is also what happens in the investment decisions. Many banks adopt the mean-variance model to invest people saves, and also suggest to their clients to not follow other intuitions. The same Gigerenzer has received an e-mail by his bank with a similar message. The fact is that the same Harry Markowitz, when he was on his way for the retirement, invested his money not using the Nobel prize winner model, but a very easy rule; the 1/N. In an interview, he said he has no regrets about his decision:

*“if the markets have profit and I am not inside these profits I will feel stupid; if the markets have losses and I am inside I will feel stupid”.*

According to Gigerenzer, this doesn't mean that the Mean-Variance model is a hoax; but that the model is an optimal model in an ideal world of well-known risks. In the real world the mathematical models used by

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<sup>18</sup> Gerd Gigerenzer, *Imparare a rischiare*, Raffaello Cortina Editore, 2015.

analysts are correct, but the risks are not known and they are unable to capture the unpredictability of the risk. As consequence of these, the “forecasts” may just work if nothing special occurs, and so it will follow the trend of the previous normal year. From a psychological aspect, when we face a complex problem our instinct is to find a complex solution to the problem. But in reality what can help us, are simple rules and simple solutions. The story doesn’t end here. In one of his congress at Morningstar, the author explained how simple rules are more favorable than complicated strategies. At the end of the congress, an analyst decided to revise all the investment of the company starting from 1969. He discovered that using the 1/N instead the other strategies, the company would have produced more profits. However ends Gigerenzer, there are still many open questions. How much should be big “N”? In which kind of stocks should it invest?

In a utopic world where the risks correspond to the assumptions of the Mean-variance models optimization is good, but in the real world can be more reliable simply and intuitive rules.

# CHAPTER 3

## APPLICATION OF THE EXPERIMENT

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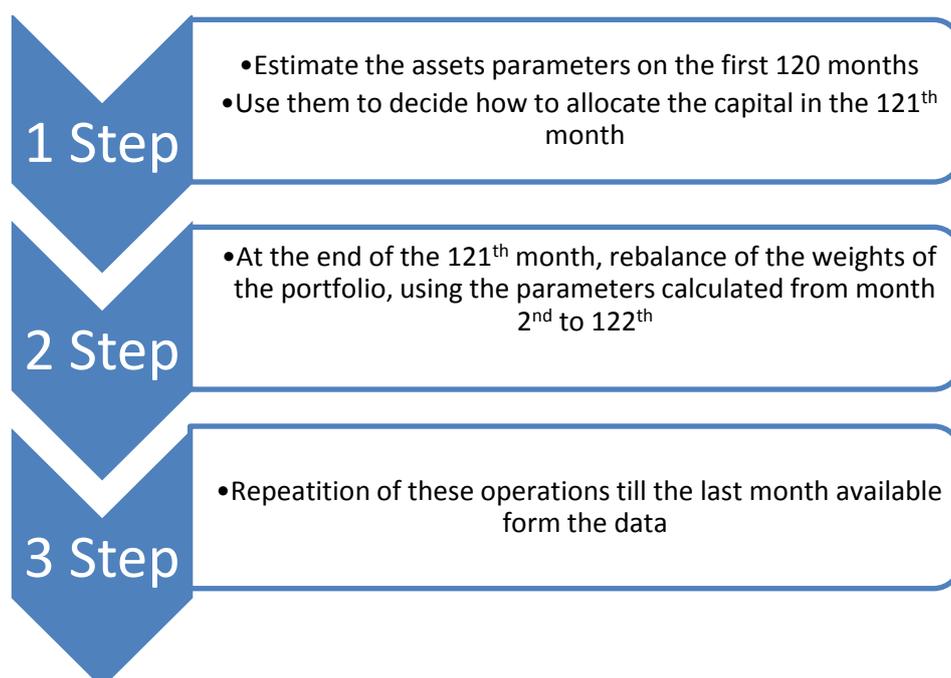
### 3.1 Introduction

In the second chapter it has been just shown an important debate that is today present in the literature; is it better a naive and heuristic strategy or is it better the more sophisticated and traditional strategy of the optimization models? These different positions are continuously persisting in the literature debate, but till now no-one has still proved the superiority of one approach respect another one. If this superiority has sometimes been temporarily shown, it has immediately been subject to many critiques and comebacks that showed the opposite, how it has been for the case of the two articles reported in the second chapter. Today it is yet not easy to answer, and we are convinced that the debate will probably last for some more time. However the previous chapter has given some important inspirations and thoughts about the portfolio models. Are simple and heuristic strategies efficient and is it possible to find among them one more reliable? Can an investor achieve a considerable return while at the same time supporting low risks? How can the estimation window affect the parameters estimation? On these premises, we are going to analyze more deeply these questions. In fact after the description of the empirical test conducted by DeMiguel, Garlappi and Uppal in which the authors declare the better performance of the  $1/N$  model over refined optimization models, and after having considered an important critique moved to such test about the criteria on which these parameters were estimated (Kritzman, Page and Turkington), we will replicate the same empirical test but using different estimation windows. In this way, we will test through a system of rebalancing portfolios how different strategies performs: a traditional model as the

“Mean Variance” that gave inspiration to many other models, a naive implementation of it, the “Minimum Variance”, and finally the naive and heuristic “1/N” model.

### 3.2 The experiment’s estimation windows

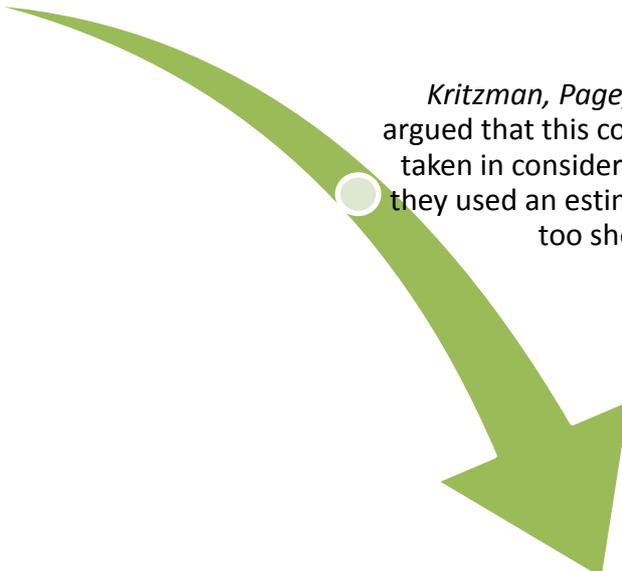
In the article “Optimal versus naive diversification: how inefficient is the 1/N portfolio strategy?” the result was critiqued because of the short estimation window used for the calculation of the parameters. As it has been already said, differently from the naive model, that does not need any “optimal” computation to decide how to allocate capital in the assets, the optimization models requires the assets moments estimation in order to allocate the capital among the available assets. This point, according to De Miguel, Garlappi and Uppal, is the reason why the optimization models should be implemented looking at the 1/N model as benchmark. The authors used in the experiment an estimation window of length equal to 120 months (10 years) to estimate the parameters, with a monthly rebalance of the portfolio. The estimation and rebalancing approach is described in the following figure.



Krtizman, Page and Turkington replicated critically to this estimation approach declaring that the estimation window used was too short, and “no thoughtful investor would blindly extrapolate historical means estimated over such short samples”. It means that 120 months are generally few to rely on (although according to DeMiguel, Garlappi and Uppal optimal portfolios are estimated on this range of time) and that the parameters obtained are not useful in this way. This intuition gave us the inspiration to replicate the experiment implementing a longer estimation window, and testing for a new hypothesis: how would the traditional and heuristic strategies perform in a system of rebalancing portfolios and with a more “reasonable” estimation window? We decided to compute the assets parameters no more on the criteria of the “120” months of the original experiment, but on 120, 150, 180 and 210 months. In the following figure is represented the new approach used.

*Demiguel, Garluppi, Uppal* declared the outperformance of the 1/N model.

*Krtizman, Page, Turkington* argued that this conclusion can'te taken in consideration because they used an estimation window too short.



## The experiment

will try to repeat the test using longer estimation windows.

### 3.3 Presentation of the experiment

At this point is possible to start replicating the experiment .The features of the new test are almost the same of the DeMiguel, Garlappi and Uppal test, but with a main important different: the time on which is built the estimation window is modified. The features are the following:

*The database:* we managed to build three of the seven datasets used in the original experiment; it was not possible to recover the other databases because they were no available. The datasets used are:

- *“International equity indexes”*<sup>19</sup>: it consists of nine international equity indexes: Canada, France, Germany, Italy, Japan, Switzerland, United Kingdom, United States and the World index. The returns are computed based on the US dollar value of the country equity index at the end of each month. The monthly returns go from: 01/1970 – 06/2001.
- *“Industry Portfolios”*<sup>20</sup>: it consists of monthly excess returns (over the risk free asset) of ten industry Portfolios in the United States (Consumer-Discretionary, Consumer Staples, Manufacturing, Energy, High Tech, Telecommunication, Wholesale and Retail, Health, Utilities, and Others) plus the market risk premium. A little variation has been made respect the previous experiment for the range of time in order to homogenize the time; the monthly returns go from: 01/1970 – 06/2001
- *“MKT, SMB, and HML portfolios”*<sup>19</sup>: They are three portfolios created by Eugene Fama and Kenneth French , and are defined by DeMiguel, Garlappi and Uppal as: “(i) MKT is the excess return on the US equity market; (ii) HML is a zero cost-portfolio that is long in high book to market stocks and short in low book to market stocks; (iii) SMB is a zero cost portfolio that is long on small caps and short

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<sup>19</sup> <https://www.msci.com/>

<sup>20</sup> [http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data\\_library.html](http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html)

in large-cap stocks”. The monthly returns go from: 01/1970 – 06/2001.

Note that in the two last databases, also if they are coming from the same source used in the original experiment, they are different from the ones originally used because the portfolio that constitute them are reformed annually, and with them also their full historical returns series.

- *Risk-free*<sup>21</sup>: the risk-free rates used in the new experiment were different according to the estimation window adopted. We used the same methodology adopted by the authors. In fact in the original experiment the risk free used was the 90 days US nominal Treasury bill; the estimation window they adopted was 60 and 120 months; in this sense, “90” was an intermediate number between 60 and 120. So, in our experiment for the estimation windows of 120 months we used the 90 days US nominal Treasury Bill; for the estimation window of 150 months, used a risk free that was obtained through a naive approach: it was the medium of the 90 days US nominal Treasury bill and the 180 days US nominal Treasury bill; for the estimation window of 180 months we used the 180 days US nominal Treasury; finally for the estimation window of 210 months we estimated a risk free using the 180 days US nominal Treasury bill and the 1 year US nominal Treasury Bill an intermediate number between 60 and 120. The monthly risk free rates go from: 01/1970 – 06/2001.

*The considered models*: one of the aims of this new experiment, was to verify how simple and heuristic strategies perform respect a more traditional and sophisticated strategy. For this reason we chose the following models:

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<sup>21</sup> <https://www.bloomberg.com/>

- Mean Variance model; it is the first optimization model that has been developed in the literature, and many other later models took inspiration from it. It can be considered a classical model. The vector of the portfolio weights is the same expressed in chapter two and used by the previous authors;
- Minimum Variance model; today is one of the most used strategy, especially after the recent financial crisis, investors have become more *risk averse* and they look for portfolios with the least possible risk. It is a derivation of the Mean Variance Model; it simply works with the covariance matrix of the assets. In this sense it can be considered in part a traditional model, but at the same time a heuristic solution of the optimization models. The portfolio weights of this model was the same of the original experiment, and it is obtained solving the following optimization problem:

$$\min_{\mathbf{w}_t} \mathbf{w}_t^\top \Sigma_t \mathbf{w}_t, \quad \text{s.t.} \quad \mathbf{1}_N^\top \mathbf{w}_t = 1 \quad .$$

- 1/N model; it is known as the heuristic and naive model for excellence, and is far away to be considered “sophisticated”. It simply divides allocate the same quantity of capital to all the assets.

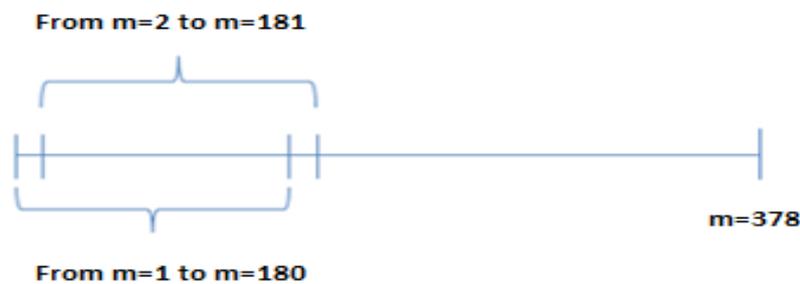
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*Measures of performance:* the strategies were evaluated under three performances of evaluation, already described in the previous chapter; they are:

- Sharpe ratio;
- Certainly Equivalent Return(CEQ);
- Turnover.

*Estimation window and rebalance:* from a moving window of 120 months we moved to a moving window equal to 120,150 180 and 210 months.

Given the dimension of the time series of the considered datasets (378 monthly information for each of the assets), it was possible to replicate respectively 258, 228, 198 and 168 iterations in the experiment for each considered estimation window (“number of information- number of months considered in the estimation window”). In this way, following the original system, the portfolio of each strategy was rebalanced monthly, using the parameters from the previous months. It follows a representation of the approach used for the estimation window of 180 months.



**Figure 20:** The approach used for the estimation window that considers 180 months.

*Statistical test:* to evaluate the different Sharpe ratio performances of each strategy across the different datasets, we used the Jobson and Korkie (1981) test revisited by Memmel in 2003<sup>22</sup>; the same was used in the original experiment.

$$\hat{z}_{JK} = \frac{\hat{\sigma}_n \hat{\mu}_i - \hat{\sigma}_i \hat{\mu}_n}{\sqrt{\hat{\vartheta}}}, \quad \text{with } \hat{\vartheta} = \frac{1}{T-M} \left( 2\hat{\sigma}_i^2 \hat{\sigma}_n^2 - 2\hat{\sigma}_i \hat{\sigma}_n \hat{\sigma}_{i,n} + \frac{1}{2} \hat{\mu}_i^2 \hat{\sigma}_n^2 + \frac{1}{2} \hat{\mu}_n^2 \hat{\sigma}_i^2 - \frac{\hat{\mu}_i \hat{\mu}_n}{\hat{\sigma}_i \hat{\sigma}_n} \hat{\sigma}_{i,n}^2 \right).$$

In details, “*i*” and “*n*” are the 2 portfolios, and their averages variances and covariances are represented by:  $\hat{\mu}_i$ ,  $\hat{\mu}_n$ ,  $\hat{\sigma}_i$ ,  $\hat{\sigma}_n$ ,  $\hat{\sigma}_{i,n}$ . “*T-M*” is the difference between the number of observations for considered “*i*” asset and the estimation window used. The test assumes that the returns

<sup>22</sup> Christoph Memmel, *Performance Hypothesis Testing with the Sharpe Ratio*, “Finance letters, (2003), no.1.

generated by the different portfolios are IID (identically and independent distributed) and the assumption is that they follow a normal distribution.

*The software:* for the computations of the experiment, we developed a code through the use of “*Matlab*”, an advanced and well known software that allows to operate different programming, statistical and financial operations. It is used in different fields, such as economy, engineering and computer science. We developed three codes “to run” in order to compute the different measures of performance of each strategy in each of the three datasets. Once the results are available, it is possible to go on with the evaluation of the results and to comment them.

### 3.3.1 Description of the “*Matlab*” code

In this subsection we present one of the codes that have been developed and used in the experiment; the code can be easily adapted to all the different databases considered. For the use of Matlab it is important to have in mind some notions of “computer programming” and to properly use some financial and statistical functions embedded in the software. Here is the description of all the steps needed to obtain the Sharpe ratio (and “its p-values), the turnover and the certainly equivalent return for all the 3 strategies (1/N, Mean Variance and Minimum Variance) for the particular “*Industry Portfolio*” database, considering the estimation window equal to 180 months.

```
clc;

load('datasets.mat')

n=198;
data=(industry_portfolio);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% method 1 over N. I defined it as method "A"
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

w_sol_a=ones(1,11)/11;
```

```
r_portrisk_a = zeros(1,n);  
r_portreturn_a = zeros(1,n);
```

**Figure 21:** Part “1” of the Matlab code.

First of all, as it is possible to look from the following picture, it is loaded the “*datasets*” file, that was created and that contains the monthly excess returns of the three different databases considered over the 180 days nominal US Treasury bill. The “*industry-portfolio*” present in the file is recalled as “*data*”. It is a matrix of monthly excess-return of dimension: “*378\*11*” (378 rows times 11 assets). With the letter “*n*” it is indicated the number of iterations that will be done in the cycle of this process. It is equal to 198; in fact the numbers of iterations is equal to the number of total observations minus the estimation window selected in the experiment (378-180=198). Together with this operation, it is created a vector of dimension 1\*11; it represents the weights invested in each asset by the 1/N strategy (called as “A” strategy). Each of these values is equal to 0.0909. Finally have been created 3 “empty” matrixes that will contain the return and the risk and of the 198 portfolios we are going to create (this programming operation is defined as *pre-allocating*).

```
for i=1:n%begining of the cycle  
    cm2_a=cov(data(179+i:180+i,:));  
    ma2_a=data(180+i,:);  
  
    [PortRisk, PortReturn] = portstats(ma2_a, cm2_a,w_sol_a);  
  
    %to each pre-allocated vector is associated the relative value  
    r_portrisk_a(i)= PortRisk;  
    r_portreturn_a(i)= PortReturn;  
  
end  
  
mediaret_a=(mean(r_portreturn_a));  
mediarisk_a=(mean(r_portrisk_a));  
sh_a=(mediaret_a/mediarisk_a);%Sharpe ratio  
disp('the Sharpe ratio of the method 1 over N is:');  
disp(sh_a);
```

```

ceq_a=(mediaret_a)-(((mediarisk_a)^2)/2);%Certainly equivalent return
CEQ_a=ceq_a;
disp('the CEQ of the 1 over N method is:');
disp(CEQ_a);

for i = 1:(n-1)%Turnover
    diffcol1_a(i)= abs(data(i+1,1) - data(i,1));
    diffcol2_a(i) = abs(data(i+1,2) - data(i,2));
    diffcol3_a(i) = abs(data(i+1,3) - data(i,3));
    diffcol4_a(i) = abs(data(i+1,4) - data(i,4));
    diffcol5_a(i) = abs(data(i+1,5) - data(i,5));
    diffcol6_a(i) = abs(data(i+1,6) - data(i,6));
    diffcol7_a(i) = abs(data(i+1,7) - data(i,7));
    diffcol8_a(i) = abs(data(i+1,8) - data(i,8));
    diffcol9_a(i) = abs(data(i+1,9) - data(i,9));
    diffcol10_a(i) = abs(data(i+1,10) - data(i,10));
    diffcol11_a(i) = abs(data(i+1,11) - data(i,11));

end

turnover_a=(sum(diffcol1_a)+sum(diffcol2_a)+sum(diffcol3_a)+sum(diffcol4_a)+sum(diffcol5_a)+sum(diffcol6_a)+sum(diffcol7_a)+sum(diffcol8_a)+sum(diffcol9_a)+sum(diffcol10_a)+sum(diffcol11_a))/(n*100);

disp('the turnover of the 1/N method is:');
disp(turnover_a);

```

**Figure 22:** Part “2” of the Matlab code.

After this preliminary operation, the cycle of 198 iterations starts. First we need to compute the expected return and the covariance matrix for the assets in the 198 iterations considered by this cycle, starting from the 181<sup>th</sup> month (operations “*cm2\_a*” and “*ma2\_a*” in the code). In fact we already have the portfolio weights that form the portfolios, and we need just to evaluate the performances of these particular portfolios composed by the equally weights. In this operation it is possible to see the use of the moving window; in fact the cycle of 198 iterations created, computes the considered values for each of the iterations, and at the same time these values are assigned to the correspondent matrixes created with the pre-allocation. After this, we have all the elements that are sufficient to compute the return and return and the risk (standard deviation) and Sharpe ratio of all the 198 portfolios created for the equally weighted

strategy. In order to obtain them, it used the *Matlab* function “*portstasts*”; it is important to specify in brackets the asset returns vector, the covariance matrix and the assets weights vector. For the given strategy it is computed the Sharpe ratio (“*sh\_a*”) and the relative certainly equivalent-return. Then it follows the strategy’s turnover.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% metodo mean variance. I defined it as method "B"
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
r_med_b=zeros(1,11);
r_cov_b=zeros(11,11);

gamma=1;
uni=ones(1,11);
r_w_sol_b=zeros(n,11);
```

**Figure 23:** Part “3” of the Matlab code.

Now the performances of the equally weighted strategy have been computed, it is considered the following strategy: the Mean Variance model (called as “B” strategy). In this case the process is a little more complicated; in fact the weights value are no more decided by a simple and heuristic solution ( $1/N$ ), but they are selected according to the model presented by the authors in the original experiment (chapter 2). So to select the optimal weights, it is important to detect and use the past values of the assets performances. Firstly, an empty vector, for the past expected return values (“*r\_med\_b*”), and empty matrices, for the past asset covariance values (“*r\_cov\_b*”) and for the optimal weights (that we are going to compute, “*r\_w\_sol\_b*”) have been created so that we will be able to collect all the data and the information we need produced by the cycle of 198 iterations. In addition to this, it has been created “*uni*”, a 11 dimensional vector of ones; it is needed to solve the problem of the “Mean Variance” model how was previously seen in chapter 2.

```
for i=1:n
```

```

cm1_b=cov(data(i:179+i,:));
ma1_b=mean(data(i:179+i,:));

if i==1
    r_cov_b=cm1_b;
else
    r_cov_b = vertcat(r_cov_b,cm1_b);
end

if i==1
    r_med_b=ma1_b;
else
    r_med_b=vertcat(r_med_b,ma1_b);
end

num=inv(cm1_b)*(ma1_b');
den=uni*inv(cm1_b)*(ma1_b');
w_sol_b=(num/den)';

if i==1
    r_w_sol_b=w_sol_b;
else
    r_w_sol_b = vertcat(r_w_sol_b,w_sol_b);
end
end

```

**Figure 24:** Part “4” of the Matlab code.

After the operation of the matrixes and of the vector pre-allocation, the cycle starts and the past values of the assets expected returns and covariance are computed. In fact, they are computed from month 1 to 180 of each asset; from month 2 to month 181; this operation till the full database is covered. The function “*vertcat*”, is used to assign the correspondent value of the parameters to the previous pre-allocated matrixes. After we obtained the estimated parameters of the assets covariance and expected return, it is possible to find the optimal weights of the investor. They are obtained solving for the vector of portfolio weights proposed by the previous authors.

```

r_portrisk_b = zeros(1,n);

r_portreturn_b = zeros(1,n);

```

```

shr_b=zeros(1,n);

for i=1:n

    cm2_b=cov(data(179+i:180+i,:));
    ma2_b=data(180+i,:);
    w_b=r_w_sol_b(i,:);

    [PortRisk, PortReturn] = portstats(ma2_b, cm2_b,w_b);
    r_temp_b = [PortRisk, PortReturn];

    r_portrisk_b(i)= r_temp_b(1);
    r_portreturn_b(i)= r_temp_b(2);

end

```

**Figure 25:** Part “5” of the Matlab code.

Now that the optimal values of the future 198 portfolios for the Mean Variance model have been obtained, it is time to evaluate their performances. As it is possible to see, the operations done are the same ones for the 1/N strategy, but this time the weights used are the optimal weights computed with the past values of expected returns and of the covariance. In the code the new optimal weights are inserted in the “*portstats*” function are the name “*w\_b*”, which come directly from “*r\_w\_sol\_b*” matrix.

```

mediaret_b=(mean(r_portreturn_b));

mediarisk_b=(mean(r_portrisk_b));

sh_b=(mediaret_b/mediarisk_b);%Sharpe ratio
disp('the Sharpe ratio of the Mean Variance method is:');
disp(sh_b);

ceq_b=(mediaret_b)-(((mediarisk_b)^2)/2);%Certainly equivalent return
CEQ_b=ceq_b;
disp('the CEQ of the Mean-Variance method is:');
disp(CEQ_b);

```

**Figure 26:** Part “6” of the Matlab code.

At this point, we have all the 198 returns and risk of the portfolio created; they are respectively in the “*r\_portreturn\_b*” and in the “*r\_port\_risk\_b*” matrixes. It is now possible to compute the Sharpe ratio of the strategy and the certainly equivalent return.

```

for i = 1:(n-1)%Turnover
    diffcol1_b(i)= abs(r_w_sol_b(i+1,1) - r_w_sol_b(i,1));
    diffcol2_b(i) = abs(r_w_sol_b(i+1,2) - r_w_sol_b(i,2));
    diffcol3_b(i) = abs(r_w_sol_b(i+1,3) - r_w_sol_b(i,3));
    diffcol4_b(i) = abs(r_w_sol_b(i+1,4) - r_w_sol_b(i,4));
    diffcol5_b(i) = abs(r_w_sol_b(i+1,5) - r_w_sol_b(i,5));
    diffcol6_b(i) = abs(r_w_sol_b(i+1,6) - r_w_sol_b(i,6));
    diffcol7_b(i) = abs(r_w_sol_b(i+1,7) - r_w_sol_b(i,7));
    diffcol8_b(i) = abs(r_w_sol_b(i+1,8) - r_w_sol_b(i,8));
    diffcol9_b(i) = abs(r_w_sol_b(i+1,9) - r_w_sol_b(i,9));
    diffcol10_b(i) = abs(r_w_sol_b(i+1,10) - r_w_sol_b(i,10));
    diffcol11_b(i) = abs(r_w_sol_b(i+1,11) - r_w_sol_b(i,11));

end

turnover_b=(sum(diffcol1_b)+sum(diffcol2_b)+sum(diffcol3_b)+sum(diffcol4_b)+sum(diffcol5_b)+sum(diffcol6_b)+sum(diffcol7_b)+sum(diffcol8_b)+sum(diffcol9_b)+sum(diffcol10_b)+sum(diffcol11_b))/n;

disp('the turnover of the Mean-Variance method is:');
disp(turnover_b);

```

**Figure 27:** Part “7” of the Matlab code.

Now, it is time to compute the turnover of the strategy. For this operation a cycle has been created that compute the different assets value among the 198 iterations for all the 11 assets.

```

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Minimum Variance method. I defined it as method "C"
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

iota=ones(11,1);

r_med_c=zeros(1,11);
r_cov_c=zeros(11,11);

```

```
r_w_min_c=zeros(n,11);
```

**Figure 28:** Part “8” of the Matlab code.

After the evaluation of the Mean Variance portfolio, it is the time of the Minimum Variance model (indicated as “C” strategy). This time, for this kind of strategy it is necessary to create a vector of ones called “*iota*”; in this way it is possible to proceed with the selection of the portfolio with the minimum variance. In fact this model, with respect to the traditional Mean Variance one, computes the optimal weights of the investors just using the past values of the assets covariance.

```
for i=1:n

    cm1_c=cov(data(i:179+i,:));
    ma1_c=mean(data(i:179+i,:));

    if i==1
        r_cov_c=cm1_c;
    else
        r_cov_c = vertcat(r_cov_c,cm1_c);
    end

    if i==1
        r_med_c=ma1_c;
    else
        r_med_c =vertcat(r_med_c,ma1_c);
    end

    w_min=((inv(cm1_c)*iota)/(iota'*inv(cm1_c)*iota))';

    if i==1;
        r_w_min_c=w_min;
    else
        r_w_min_c = vertcat(r_w_min_c,w_min);
    end
end
```

**Figure 29:** Part “9” of the Matlab code.

Then, it is possible to compute the past asset covariance values and to find the optimal weights through the solution previously provided in the description of such model.

```

r_portrisk_c = zeros(1,n);

r_portreturn_c = zeros(1,n);

shr_c=zeros(1,n);

for i=1:n
    cm2_c=cov(data(179+i:180+i,:));
    ma2_c=data(180+i,:);
    w_c=r_w_min_c(i,:);

    [PortRisk, PortReturn] = portstats(ma2_c,cm2_c,w_c);
    r_portrisk_c(i)= PortRisk;
    r_portreturn_c(i)= PortReturn;

end

```

**Figure 30:** Part “10” of the Matlab code.

Now that we have the weights derived from the portfolio with the Minimum Variance, it is possible to compute the returns and risk of the 198 portfolio created.

```

mediaret_c=mean(r_portreturn_c);

mediarisk_c=mean(r_portrisk_c);

sh_c=mediaret_c/mediarisk_c;%Sharpe ratio
disp('the Sharpe ratio of the Minimum Variance method is:');
disp(sh_c);
ceq_c=(mediaret_c)-(((mediarisk_c)^2)/2);%Certainly equivalent return
CEQ_c=ceq_c;
disp('the CEQ of the Minimum Variance method is:');
disp(CEQ_c);
for i = 1:(n-1)%Turnover
    diffcol1_c(i)= abs(r_w_min_c(i+1,1) - r_w_min_c(i,1));
    diffcol2_c(i) = abs(r_w_min_c(i+1,2) - r_w_min_c(i,2));
    diffcol3_c(i) = abs(r_w_min_c(i+1,3) - r_w_min_c(i,3));
    diffcol4_c(i) = abs(r_w_min_c(i+1,4) - r_w_min_c(i,4));
    diffcol5_c(i) = abs(r_w_min_c(i+1,5) - r_w_min_c(i,5));
    diffcol6_c(i) = abs(r_w_min_c(i+1,6) - r_w_min_c(i,6));
    diffcol7_c(i) = abs(r_w_min_c(i+1,7) - r_w_min_c(i,7));

```

```

diffcol8_c(i) = abs(r_w_min_c(i+1,8) - r_w_min_c(i,8));
diffcol9_c(i) = abs(r_w_min_c(i+1,9) - r_w_min_c(i,9));
diffcol10_c(i) = abs(r_w_min_c(i+1,10) - r_w_min_c(i,10));
diffcol11_c(i) = abs(r_w_min_c(i+1,11) - r_w_min_c(i,11));
end
turnover_c=(sum(diffcol1_c)+sum(diffcol2_c)+sum(diffcol3_c)+sum(diffcol4_c)+sum(diffcol5_c)+sum(diffcol6_c)+sum(diffcol7_c)+sum(diffcol8_c)+sum(diffcol9_c)+sum(diffcol10_c)+sum(diffcol11_c))/n;
disp('the turnover of the Minimum Variance method is:');
disp(turnover_c);

```

**Figure 31:** Part “11” of the Matlab code.

Finally the Sharpe Ratio, certainly equivalent return and turnover of the strategy are computed. The process is the same of strategy “B”, but only the names of the variables used change.

```

%%%%Statistical test: Mean Variance and 1/N%%%
jb1=jbtest(r_portreturn_a);
jb2=jbtest(r_portreturn_b);
if (jb1==0) & (jb2==0)
    Rac = horzcat(r_portreturn_a',r_portreturn_b');
    beta=zeros(197,2);
    r_cov_beta2=zeros(198,1);
    r_var1=zeros(198,1);
    r_var2=zeros(198,1);
    for i=1:(n-1)
        beta=Rac(i:i+1,:);
        cov_beta = cov(beta);
        qq=cov_beta(1,2);
        var1=sqrt(cov_beta(1,1));
        var2=sqrt(cov_beta(2,2));
        if i==1
            r_cov_beta2=qq;
        else
            r_cov_beta2=vertcat(r_cov_beta2,qq);
        end
        if i==1
            r_var1=var1;
        else
            r_var1=vertcat(r_var1,var1);
        end
        if i==1
            r_var2=var2;
        else
            r_var2=vertcat(r_var2,var2);
        end
        end
    mcov=mean(r_cov_beta2);

```

```

mvar1=mean(r_var1);
mvar2=mean(r_var2);
h1=(mean(Rac(:,1)));
h2=(mean(Rac(:,2)));
l1=mvar1;
l2=mvar2;
o=(h1*h2/l1*l2)*(mcov^2);
tetavalue=((2*(l1^2)*(l2^2)-2*(l1*l2)*(mcov)+0.5*((h1^2)*(l2^2))+0.5*((h2^2)*(l1^2))-
o))/198;
z_an=((l2*h1)-(l1*h2))/sqrt(tetavalue);
disp('final p-value');
disp(z_an);
else
disp('Mean Variance-1/N test:');
disp('normality assumption rejected');
end

```

**Figure 32:** Part “12” of the Matlab code.

Now that all the measures of performance have been computed, it is possible to replicate the statistical test of Memmel (2003), verifying if the difference between the Sharpe ratios of the optimal models and of the naive strategy are statistically significant. The test, assumes that the returns generated by the portfolio created for each strategy follow a normal distribution, otherwise it is not possible to apply the test. In order to verify this condition, we used the Jarque Bera test (“*jbtest*”) on the returns. This particular test looks for the kurtosis and the symmetry of the data, and on the basis of them it assumes or rejects the hypothesis of normality. We applied the test through the use of a *conditionl* statement; in fact if the returns of the considered portfolios follow a normal distribution the test will be executed, and the p-value will be computed; otherwise it will not be possible to compute the p-value because the assumption of normality was rejected. Here is reported the test provided in the experiment. The same has been done for the comparison of strategy “A” and strategy “C”. Now that we have computed all the performances, it is possible to run the code. After clicking the “run” button, the monitor will display the results obtained.

### 3.5 Article results

In the original experiment, the results for the three models considered under the three different measures of performance are represented in the following tables:

STRATEGY \ DATABASE	"Industry Portfolios"	"International indexes"	"MKT,SMB,,HML"
1/N	0.1353	0.1277	0.2240
Mean - Variance	0.0679 (0.17)	-0.0332 (0.03)	0.2186 (0.46)
Minimum variance	0.1554 (0.30)	0.1490 (0.21)	0.2493 (0.23)

**Table 6:** Sharpe ratio results for the considered models.  
**Source:** Look at [9].

#### Sharpe ratio Comments:

- 1/N vs Mean Variance: in this case, comparing the Sharpe ratio results for the two strategies, the naive strategy is clearly performing better than the Mean Variance. According to the authors, this is clearly connected to the estimation errors of the parameters, which lead the wrong allocation of the capital. This explains the bad result of the strategy.
- 1/N vs Minimum variance: at a first sight, the Minimum Variance seems to outperform the equally weighted model. In reality the "p-value" computed on the differences of these Sharpe ratios are not statistically significant.

STRATEGY \ DATABASE	"Industry Portfolios"	"International indexes"	"MKT, SMB, HML"
1/N	0.0216	0.0293	0.0237
Mean - Variance	606594.36	4475.81	2.83
Minimum variance	21.65	7.30	1.11

**Table 7:** Turnover results for the considered models.  
**Source:** Look at [9].

*Turnover* comments:

- 1/N vs Mean –Variance: the turnover of the equally weighted strategy is practically equal to “0”. For the Mean Variance strategy the result is completely opposite and, except for the database “MKT, SMB, HML” (that is composed by three assets), it has got the higher turnover. This is due to the estimation error problem. In a real world, this would have substantial and high cost on the strategy because of the transaction costs.
- 1/N vs Minimum variance: in this case the turnover of the Minimum Variance is higher than the one of the naive strategy. However, it is important to say that with respect to the Mean Variance model, this optimization models clearly outperforms it. This is evident looking at the results achieved by the two models in this kind of performance. The reason must be found in the how the two strategies compute the optimal weights; the Mean Variance model considers both the assets expect returns and the assets covariance matrix, while the Minimum Variance considers just the covariance matrix. This means that

considering also the estimates of the past assets returns brings to a bigger estimation error that affects the performances of the model. In this way the Mean Variance model results to be more complicated and more sensitive to the estimation error with respect to the Minimum Variance model.

STRATEGY \ DATABASE	"Industry Portfolios"	"International indexes"	"MKT,SMB,,HML"
1/N	0.0050	0.0046	0.0039
Mean - Variance	-0.7816	-0.1365	0.0045
Minimum variance	0.0052	0.0054	0.0039

**Table 8:** CEQ results for the considered models.  
**Source:** Look at [9].

CEQ comments:

- 1/N vs Mean Variance: the equally weighted strategy also in this case clearly outperforms the traditional Mean Variance Portfolio. In addition to this, it is possible to see that in two cases the optimization model has a negative CEQ. This highlights the bad performance of the strategy.
- 1/N vs Minimum variance: the Minimum Variance achieves a higher CEQ, but this data is not sufficient to explain its superiority, linked also to the statistical insignificance of the Sharpe ratios (look at the  $p$ -value previously computed).

The evaluation of the performances shows the superiority of the 1/N strategy over the two optimization models. Remember the estimation

window considered was 120 months. Should things change with a longer estimation window as suggested by Kritzman, Turkington and Page?

### 3.6 Experiment results

The results obtained from the experiment using the different estimation windows of 120, 150, 180 and 210 months are substantially very similar and they show that increasing the number of months in the estimation window used allows to all the strategies to have better performances with respect to an estimation window with fewer months. We decided to report the results for the estimation window of 180 months. It is possible to find the results for all the estimation windows in the appendix.

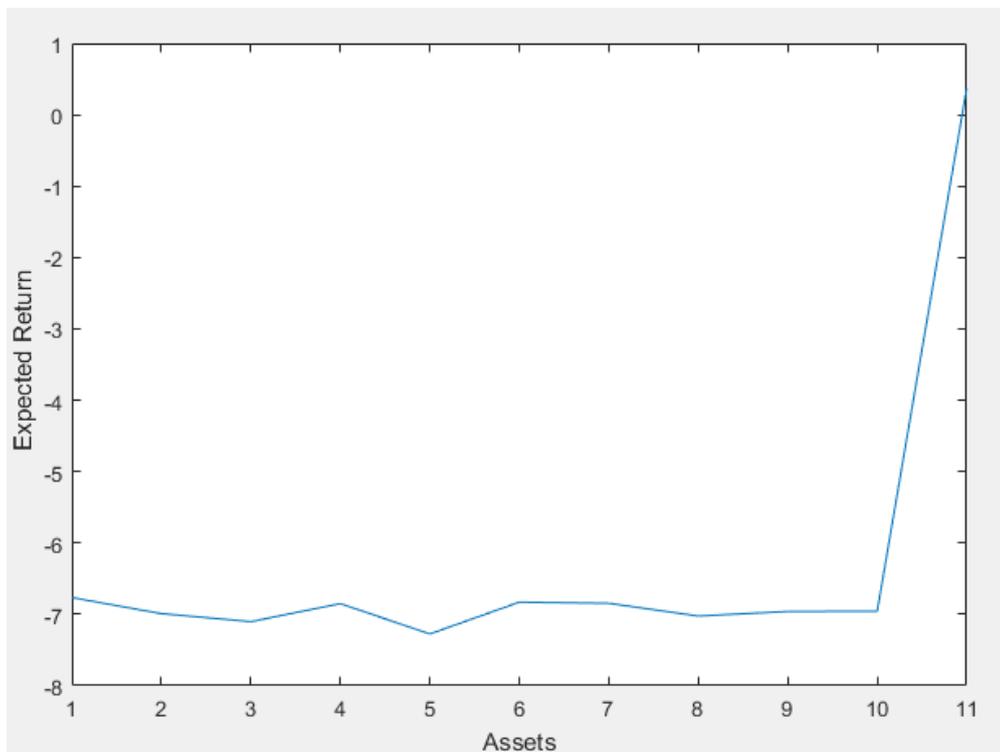
<div style="text-align: right;">DATABASE</div> <div style="text-align: left;">STRATEGY</div>	<i>"Industry Portfolios"</i>	<i>"International indexes"</i>	<i>"MKT,SMB,,HML"</i>
<i>1/N</i>	-1.2430	-1.4193	-3.6592
<i>Mean - Variance</i>	<b>2.3562</b>	<b>-1.3963</b>	-4.5194
<i>Minimum variance</i>	0.4002	-1.5076	<b>-3.0276</b>

**Table 9:** Sharpe ratio results for the new experiment.  
**Source:** Own production.

*Sharpe ratio* comments: looking at the Sharpe ratio results, it is possible to observe that the new estimation window adopted had a great impact on the strategies performances. In fact, it cannot be said as previously that the equally weighted strategy is the best performing, but for each of the considered datasets, there are different results. First of all, in the *"Industry Portfolios"* database, the Mean Variance and the Minimum

Variance Portfolio clearly outperform the negative Sharpe ratio of the naive strategy. This can be explained by the fact that one of the assets considered had a large number of positive returns (the “ $R_m - R_f$ ” assets) and so the optimization models, which take in account the risk-aversion of the investor, invested in this asset. The 1/N strategy instead could not adopt this particular allocation, and it simply allocated the same part of capital to all the assets. In the “*International indexes*” the Mean Variance is still the best performing model, followed by the 1/N model and finally the Minimum variance model. In the last databases, the hierarchy still changes; in fact this time the Minimum Variance has the highest Sharpe ratio, and the equally weighted and the Mean Variance strategies follow it.

Analyzing more deeply the “*Industry Portfolios*” database, it is possible to observe that all the first ten assets had a negative expected return values, while the last asset presented a positive average return. As consequence of this, the optimization model will invest most of their capital in this asset.



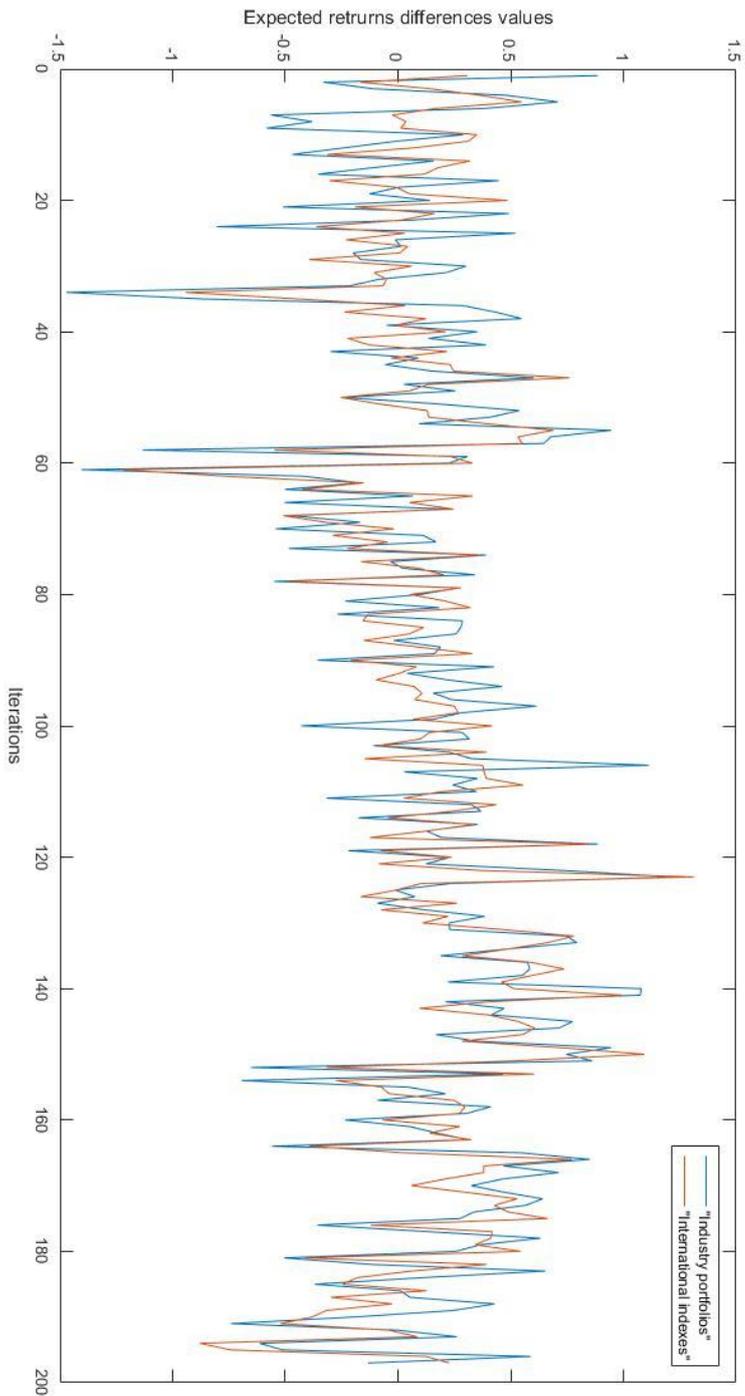
**Figure 33:** Assets expected returns values for the *Industry Portfolios* database.

STRATEGY \ DATABASE	"Industry Portfolios"	"International indexes"	"MKT,SMB,,HML"
1/N	0.6040	0.5587	0.1060
Mean - Variance	164.6506	<b>0.2143</b>	0.0228
Minimum variance	<b>0.1161</b>	11.8616	<b>0.0117</b>

**Table 10:** Turnover results for the new experiment.  
**Source:** Own production.

*Turnover* comments: analyzing the turnover results, the Minimum Variance model has an average turnover lower than the Mean Variance model, while the 1/N has performed a result that was never able to outperform the optimization models. However, the turnover of this strategy never results in high values such as "164.6506" for the Mean Variance model in the first database, or "11.8623" of the Minimum Variance in the second database. In addition, the equally weighted strategy resulted in the worst turnover in the third database. Looking specifically at the optimization models, in the first database the Mean Variance has a very high turnover compared to the other optimization model; in the second database instead the Minimum Variance model achieved a higher turnover. This result can probably be explained by the variation of the expected return assets values for the Mean Variance model, while for the Minimum Variance model the only explanation can be found in the covariance assets value (because it is the only input it needs to work). For the first optimization model, we decided to analyze the differences of the assets expected returns value in each of the

iterations for the two first databases. According to these values (that most influence the weight allocation) the assets weights were computed. The result is reported in the following picture.



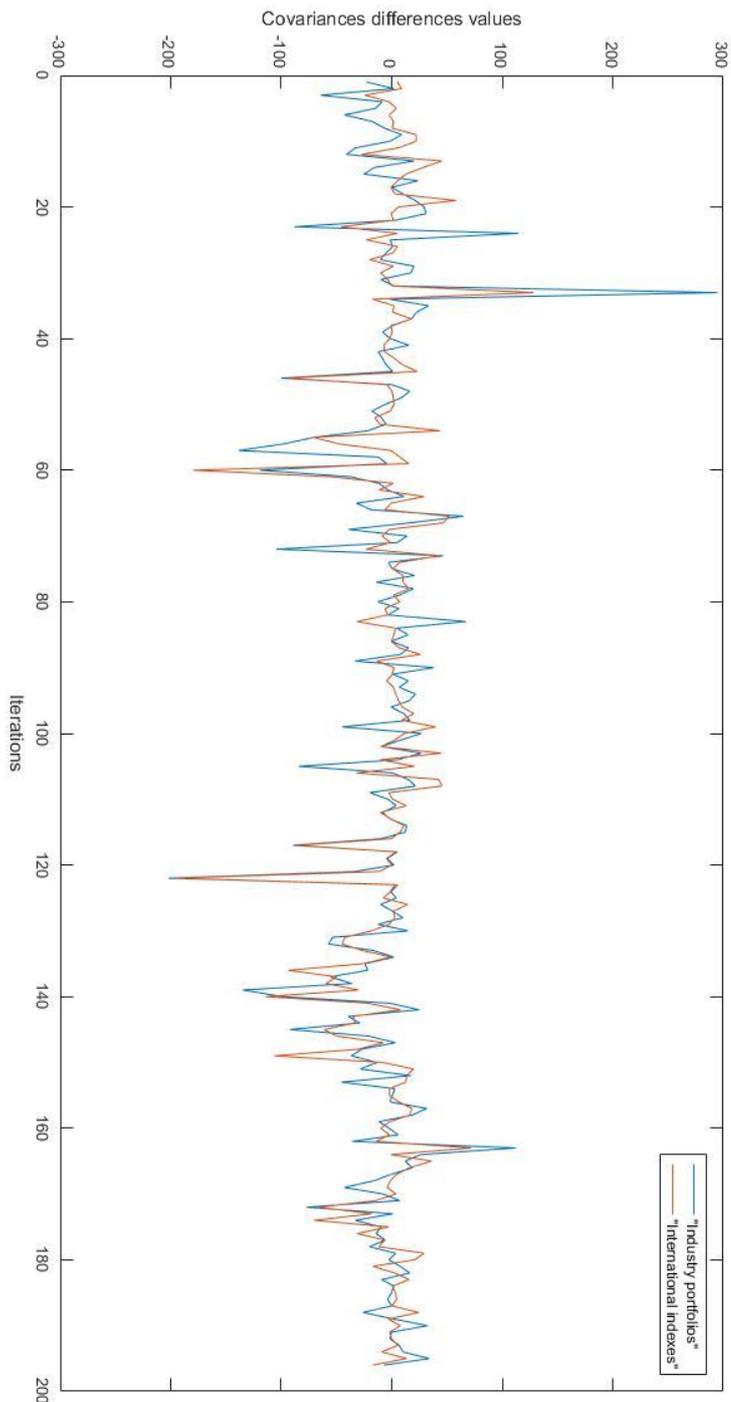
**Figure 34:** Assets expected returns differences values for the "Industry portfolios" and "International countries" databases.

As it is possible to see in the picture, for the “*Industry portfolios*” database (reported in blue line), has got higher fluctuation than the “*International indexes*” database. This means that the assets values change a lot between each iteration, and the portfolio is forced continuously to readopt in order to find the optimal portfolio. This automatically leads the portfolio to have a higher turnover for the first database.

For the Minimum Variance model, the reason of the higher turnover in the second database should be found in the covariance matrixes values of the two databases. In particular, we would expect to find higher positive values for the second database, because if there are few negative correlation values between the assets, the portfolio would have to be revised frequently to find a better one. Also in this case, since in the literature was not found anything helpful for the analysis of the result, we developed a rough but efficacious way; we computed the differences on matrixes values between each iteration, and we summed the elements of each matrix that we obtained. In this way we obtained a value that gave us an idea about how the assets covariance matrixes differed among the iterations. The result is proved in the following picture in the next page. The first database seems to have a larger number of negative values computed on the differences of the matrixes values, and this according to the previous reasons would suggest that has got a lower turnover respect the second database. This is confirmed by the experiment results.

In the last database both the two optimization strategies showed a very low turnover; this is can be explained by the few number of assets that are present in such database (it is composed of three assets). From the turnover analysis, the heuristic strategies have achieved on average a low turnover; in real investment this would translate in low transaction costs. For the more sophisticated and traditional strategy instead, in the first database where previously the model achieved the better Sharpe ratio,

had a substantial turnover; in a real investment this could have a great impact on the final cost of it, and it may erode all the possible gains earned by this optimization model.



**Figure 35:** Assets covariance differences values for the "Industry portfolios" and "International countries" databases.

STRATEGY \ DATABASE	"Industry Portfolios"	"International indexes"	"MKT,SMB,,HML"
1/N	-9.2026	-10.2911	-4.0342
Mean - Variance	-5008	-9.9754	-7.1675
Minimum variance	<b>-3.2183</b>	<b>-9.7886</b>	<b>-3.7718</b>

**Table 11:** CEQ results for the new experiment.  
**Source:** Own production.

CEQ comments: finally, it is time to evaluate the last measure of performance considered: the certainly equivalent return. The "CEQ" expresses the risk free return the agent is ready to accept rather than following that particular strategy. It is useful to consider because it can tell us something more about the results previously obtained. The large presence of negative CEQ present in the table is due to the fact that most of the strategies had negative expected returns, as was confirmed by the Sharpe ratio results. In that kind of performance, usually the optimization models outperformed the equally weighted strategy. An apparently good result was achieved by the Mean Variance model in the "Industry Portfolio" database. But in this case, the CEQ shows us that the investor would accept a high negative return as alternative to enter in that strategy; this because although the strategy generated a positive return, the risk that the strategy supported was very high. So in this sense the certainly equivalent return integrate the previous results. It is possible to

see that the heuristic strategies achieved good results, outperforming the Mean Variance model (especially the Minimum Variance).

### **3.6 Experiment conclusions**

The experiment we just showed, presented a system of rebalancing portfolio in which heuristic and traditional strategies were performed. In particular, it was important to judge the three models under three measures of performances. In fact analyzing the performances just in terms of Sharpe ratio, as Kritzman, Page and Turkington did in their experiment, it would have lead to a wrong conclusion: the outperformance of the Mean Variance model, followed by the Minimum Variance. But starting from the fact that the experiment included a monthly rebalance, it was opportune to consider the turnover of the strategies, in order to asses a potential impact of the rebalancing, and in order to better interpret the Sharpe ratio results, also to compute the CEQ of the strategies. . As first result, it is clear that heuristic strategies are able to compete with the more traditional and sophisticated Mean Variance.

At the same time, we replicated the test of the previous authors implementing a longer estimation window to allow to the optimization models to rely on a longer data sample. Of course these results cannot be directly compared in terms of values with the result of the original experiment, because as already said, the database considered are continuously updated with new portfolio structures, and with them the historical performances of such securities. However the Mean Variance optimization model, previously critiqued and “defeated” by the naive strategy, showed performances that were not so far from the ones of the Minimum Variance and of the 1/N strategy. In this sense, adopting a longer estimation window improved the performances of the optimization models. Analyzing more deeply the performances of each strategies, the 1/N model achieved in general good results in the CEQ and Sharpe ratio terms, and for the turnover measure it resulted to be the

best model on average. However, one of the main limits that emerged is the fact that in a situation in which one of the assets of the database selected had large positive returns (the " $R_m - R_f$ " asset in the "*Industry portfolios*"), the strategy just allocated the same fraction of capital to all the assets, performing in this way a negative Sharpe ratio against the positive values of the traditional optimization models. In this way the strategy appears to be very strict, unable to consider the risk aversion of the investor and investing in the asset with more positive expected return. In the other two databases however the strategy achieved a better result, also if it never resulted to be the best model. For the turnover performance the strategy registered very good results, also if in the last database "*MKT, SMB, HML*" (that included 3 assets) it registered the worst performance (however always a good result if compared to the maximum turnover values of the other strategies).

The optimization models benefitted of the new estimation window, with the Mean Variance that achieved similar results of the heuristic strategies, while the Minimum Variance resulted to be the best strategy in some cases. At first sight, looking especially at the Sharpe ratio results, the Mean Variance model could seem to be the best optimization strategy; in fact in the "*Industry portfolios*" and "*International Indexes*" databases, the models had values : "2.3562" and "-1.3963" , against the values : "0.4002" and "-1.5076" of its rival. Just in the "*MKT, SMB, HML*" database the Minimum Variance performed a higher Sharpe value: "-3.0276" against "-4.5194". However these result could not be interpreted alone; in fact the results should be considered as a whole, so taking in consideration also the other two measures of performance: the turnover and the certainly equivalent return of the strategy. From the analysis of the certainly equivalent return, it is evident that although the Mean Variance could perform a high Sharpe ratio, because of a high expected return, the risk supported by the investor is very high, as in the situation of the "*Industry portfolios*" database; in fact the investor would prefer to

accept a certain negative return of “-5005” percent, against the “-3.2183” percent of the Minimum Variance model. Also in the other 2 databases the latter models performed a higher CEQ, especially for the “*International indexes*” where the former models had a higher Sharpe ratio. Finally also for the turnover performance the Minimum Variance model had an average performance lower than the Mean Variance.

In conclusion, the experiment has shown that in a system of rebalancing portfolios and where the assets parameters are computed on longer estimation windows with respect to the traditional ones , the heuristic models are able to compete (and have higher performances) than a traditional and sophisticated strategy. In addition to this, it would seem that more past data are considered in the estimation window, the better the considered strategies perform. The Minimum Variance model has also shown that it is possible for the investor to achieve good investment performances (in terms of Sharpe ratio, turnover and CEQ) supporting a low investment risk.

# “CONCLUSIONS”

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In the financial world many different models are present that can be useful for the investor. Every time a new model arises, it is first of all a new tool for the investor and for the literature. In fact it could be a new starting point for the previous models to may implement themselves and at the same time it could give new ideas and inspiration for the future models. In this sense, it would be silly to say that there are models that should be completely eliminated or useless.

The existing literature, presents a large part of optimization models which are born with the first founder of the portfolio theory, Harry Markowitz, but not for this all models that are successively born have used the same principles of the Mean Variance optimization models. Today are present models that rely on an equally weighed composition, models that allocate a determined part of the capital in fixed parts (like the 60/40 or the 1/N models), or models that are based bees swarm. Financial models are continuously renewing themselves and they compete to be the best financial portfolio model. The research is really alive!

As seen in this thesis, every model has got it pros and cons; today it is impossible to state the outperformance of a single model respect another one. A model could be able to captures all the factors determining the choices and preferences of the investor but at the same it could be so sophisticated that a small error could have a big impact. Maybe it is possible to test them and to compare their performance, highlighting the strong and weak point, but it is more difficult to state the absolute outperformance of a certain model. Most of the time the results obtained are related to the criteria adopted in the test (as the time estimation window adopted in our experiment) or to the specific range of time of the economy considered. The variables that influence the result are many

and they differ between each other. In this sense the results are not always comparable.

In the thesis we wanted to conduct a test with the aim to see if heuristic and simple strategies (1/N and Minimum Variance models) are able to compete with a more sophisticated and traditional strategy (Mean variance model) in a system of rebalancing portfolios. The Mean Variance optimization model of Harry Markowitz can be considered as the “father” of the optimization models and many other models that took inspiration from it. One of these is the Minimum Variance model, a simpler “version” of its previous predecessor. Together with these models, it was considered the equally weighted strategy, known also as the “1/N” model. It is a completely different alternative to the optimization models, which, although very naive and simple, has found consideration among the investors. The choice of these models has shown what we said in the previous sentences: from one side the traditional and consolidated models can change and implement themselves, becoming also simpler and less complex, and on the other side that new alternative modes can base on different principles. All these models present some pros and cons; the Mean Variance and the Minimum Variance models are strongly based on the assets past values (that involves the presence of the estimation error), while the equally weighted strategy does not rely on none information (would you ever invest in Government bond without considering if in the past the country was in default?).

The experiment in this case has shown that heuristic and simple solutions are able to compete and to offer a good alternative to a more sophisticated strategy. Of course the experiment confirmed that both the two heuristic solutions have some disadvantages; the Minimum Variance is in part linked to past values of the assets and this can affect its performances (as for the turnover in the second database); the 1/N cannot exploit some notions of the markets, especially the past notions and maybe some extra notions like some future beliefs (as for the Sharpe

ratio in the first database). With this sentence we do not want to say that the future should be strictly linked to the past, but it could be helping for some future beliefs. A part from this, the equally weighted strategy did not perform so poorly in our experiment, such that it didn't result to be the "worst" strategy. This result could mean that when the investor is in front of unknown opportunities, he could think to apply a naive strategy (like in some IPO opportunities, when the company is for the first time entering in the stock market). In this of course way he would protect himself adopting a good diversification, and at the same time the strategy could generate some profits. However, if the investor would prefer to choose according to his beliefs or to some extra notions, the investment could be more risky; from one side he could generate profits, on the other side investing in something still unknown according to his beliefs or to some external (and maybe misleading?) information could be a not good investment and he could be too much exposed. In this way the  $1/N$  strategy avoids high concentrated positions and so the investor will never be too much in a risky position. On the other side, among the optimization models, the Minimum Variance model resulted to be able to perform better than the Mean Variance model, supporting a lower risk. These results just proved by the experiment showed that heuristic and simple strategies are able to outperform a sophisticated strategy; the investor can rely on them. It is difficult to identify a unique general better model among the two heuristic solutions. One could be good to be adopted when there is a long data history available of the assets considered, the other when the investment opportunities that arise are new. In all the cases, the best portfolio strategy will never be something equal for each investor, for each kind of investment and for each time of the economy, but will be always something adequate to the situation the investor is facing. Another important aspect of the experiment was the realization of a multi periodical strategy, able to revise the weights selected monthly. This aspect is very important, because it is true that the

transaction costs today are cheaper, but they are still present and able to affect the different portfolio strategies. In this case the strategy was able to give a rough idea of the amount of trades needed for each strategy. In this way the system of rebalancing portfolios did not result to be too much sophisticated, but efficacious.

A future possible aspect to consider for the future heuristic solutions would be to try to considerate also the taxation on the capital gains. Of course this would mean to create more complex and sophisticated models, but how has been for the models present today, starting on this direction will help in the future the models to get more realistic and, after some probably implementations and revision, they could be easily incorporated in traditional portfolios models.

# APPENDIX

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In this appendix are reported all the results of the application of the experiment for all the estimation windows considered. How already reported in chapter 3, it is possible to see that the performances of strategies increase with the use of longer estimation windows, and that the simple strategies seem to be competitive with respect to a more traditional and sophisticated strategy.

### List of abbreviation

M=Estimation window

1/N=Equally weighted strategy

Mv= Mean Variance model

Min=Minimum Variance model

p="Industry portfolios"

c="International indexes"

f="MKT, SMB, HML"

- Sharpe ratio results for the experiment:

### ESTIMATION WINDOW

	M=120	M=150	M=180	M=210	
<b>1/N</b>	-1,5166	-1,3188	-1,243	-1,3179	<b>p</b>
<b>Mv</b>	3,851	-4,0057	2,3562	-4,8991	<b>p</b>
<b>Min</b>	0,4975	-3,2691	0,4002	0,4132	<b>p</b>
<b>1/N</b>	-1,7709	-1,5557	-1,4193	-1,5129	<b>c</b>
<b>Mv</b>	-1,7139	-1,5105	-1,3963	-1,4704	<b>c</b>
<b>Min</b>	-1,8226	-1,6414	-1,5076	-1,544	<b>c</b>
<b>1/N</b>	-4,0858	-3,8396	-3,6592	-3,5718	<b>f</b>
<b>Mv</b>	-4,8904	-4,7359	-4,5194	-4,2734	<b>f</b>
<b>Min</b>	-3,4291	-3,2135	-3,0276	-3,1282	<b>f</b>

STRATEGY

DATABASE



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