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Dependence in Extreme Events
An analysis based on Copula Functions

Supervisor
Ch. Prof. Davide Raggi

Graduand
Jacopo Vecchio
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Introduction

In modern finance world, normality represents a concept that is fundamental to shape our expectations through financial modeling. However, often this assumption fails to realistically represent financial markets, which frequently show non-normality features. Recent periods were characterized by extreme events: European debt crisis, covid19 and Ukraine war jeopardized financial markets stability. Normality assumption is not suitable for this kind of events that are characterized by non-linearity, fat tails and volatility clusters. The impact by negative shocks on financial markets obviously lead to portfolio losses. In this thesis we are interested in discovering a better way to describe financial world data to minimize risk. As stated before, normality assumption is not an effective solution to shape extreme events and their effects on financial firms. Thus, we need more sophisticated tools to effectively analyze financial data. Through this thesis we show different methods that allow to study extreme events. From extreme value theory to copula function, our aim is to establish which kind of model better describes price movements between two stock indexes and how the two are related. In our analysis we need to find an optimal fit for the stock indexes and determine marginal distributions for constructing copulas. We choose two stock indexes since they are formed by many companies. CAC 40 and FTSE MIB respectively represent French and Italian most important stock index.
Firstly, we verify if our stock indexes follow either a normal or student's t distribution, then we analyze Extreme value theory fit to our data. Once we established optimal marginal distributions we can build copulas on our findings and plotting optimal risk measures. Copula functions represent a way to study relationship between price movements of financial products and how deep is the influence of one over the other.

Concluding our thesis, we summarize various risk measures introduced during our dissertation and add a new concept: stress testing. By merging risk measures and stress test we realize a complete portfolio management system. For instance, a kind of risk measure that is computed when considering stress test is the probability of experiencing a simultaneous loss in a portfolio composed by CAC 40 and FTSE MIB.

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1. EVT: Block Maxima and Peaks over Threshold

Extreme Value Theory (EVT) focus on studying the behavior of tails in random variables. In financial market, a tail event could reveal disruptive and threatening towards market stability. In statistics these events are usually considered as higher quantile of the loss distribution. Our study is an example of extreme and possible risk measures to account for it. Applying Extreme Value theory in risk management allows to investigate what can be the impact of a relative rare event and model the possible outcomes.

In this paragraph we are going to analyze Extreme value theory following two different approaches: Block Maxima Method and Peaks Over Threshold.

The Fisher-Tippet theorem describe the probabilistic behavior of maxima and random variables in terms of Generalized Extreme value distribution:

"Consider a sequence of i.i.d. random variables $x_1, x_2, \dots, x_n$,, representing risks or losses with an unknown cumulative distribution function F(x) = Pr(X_i ≤ x). For financial applications, X_i describes negative returns (or losses).

$$\lim_{n\to\infty} Pr\left(\frac{M_n - \mu_n}{\sigma_n} \le x\right) = \lim_{n\to\infty} F^n(\sigma_n x + \mu_n) = H(x)$$

for a non-degenerate distribution function $H(x)$. If this condition holds, F is said to be in the maximum domain of attraction of H, and H is of type H_ξ, namely, a Generalized Extreme Value (GEV) distribution

$$H_\xi(z) = \begin{cases} exp\left\{-(1+\xi z)^{-1/\xi}\right\} & \xi \neq 0, 1+\xi z > 0 \\ exp\left\{-exp\left\{-z\right\}\right\} & \xi = 0, -\infty \le z \le \infty \end{cases}$$

then H is of the type H_ξ, for some parameter ξ's value."[1]

In this case μ_n and σ_n can be considered as location and scale parameters, whereas ξ is a shape parameter that determines the tail behavior of H_ξ. In particular, ξ determines the shape of H_ξ resulting in three different distributions type:

- ξ<0: H_ξ is a Weibull type distribution;
- ξ=0: H_ξ is a Gumbel type distribution;

[1] Viviana Fernandez, extreme value theory: value at risk and returns dependence around the world

Axel Bucher (2018), An overview of nonparametric tests of extreme-value dependence and of some related statistical procedures

- ξ>0: H_ξ is a Fréchet type distribution.

Weibull type distribution differentiates itself from the other type having finished tails. In fact, Gumbel type presents thin tails while Frechét type is characterized by thick tails. We are particularly interested in Frechét type since usually returns distributions are leptokurtic. This is particularly true if we analyze riskier assets such as stocks.

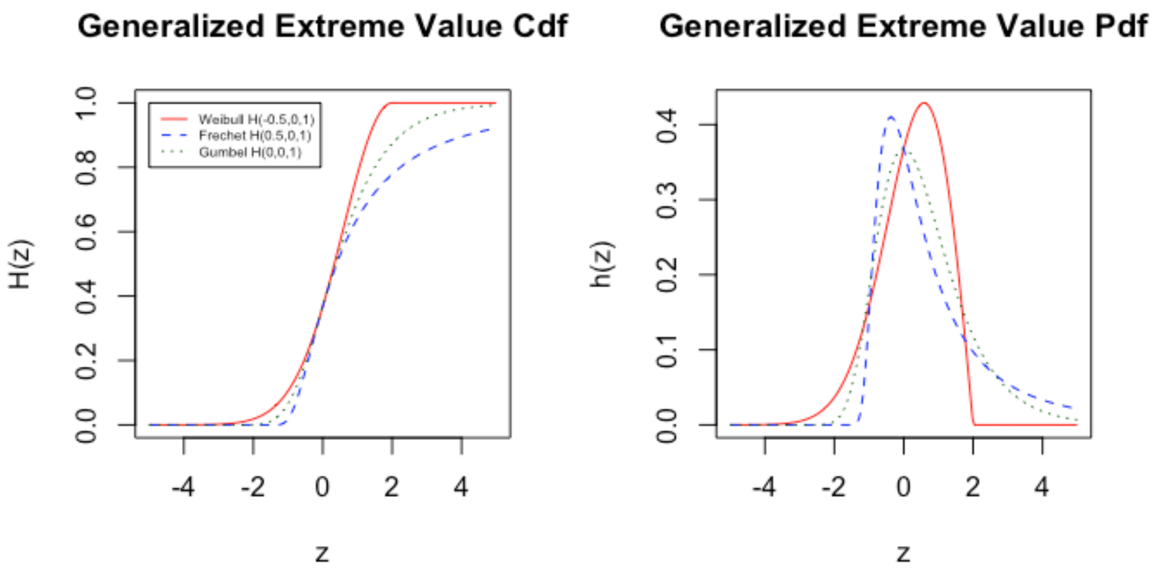

Figure 1 Cumulative and Probability distribution functions for the GEV variable, with μ=0 and σ=1.

Comparing Block Maxima and Peaks Over Threshold

To apply the Fisher-Tippet theorem we need to estimate μ, σ and ξ. This cannot be possible if we focus just on a unique maximum, to solve this issue we implement Fisher-Tippet theorem either through Block Maxima or through Peaks Over Thresholds (POT). The former consists of dividing the dataset into equal length blocks and finding a maxima for each of them. This mechanism is based on two principles that can be contradictory: to guarantee convergence to GEV we need large blocks. Nevertheless, we need a large amount of block ending up having many observations. In fact, this lead to have lower variances of Maximum likelihood estimates. The likelihood for these “local” maxima is written as:

$$\ell(\mu,\sigma,\xi \mid \mathbf{M}) = - \quad m\ln(\sigma) - (1+1/\xi)\sum_{i=1}^{m} \ln\left[1+\xi\left(\frac{M_n^{(i)}-\mu}{\sigma}\right)\right]$$
$$-\sum_{i=1}^{m}\left[1+\xi\left(\frac{M_n^{(i)}-\mu}{\sigma}\right)\right]^{-1/\xi}$$

As seen before, we have another trade-off between Maximum Likelihood Estimator bias and variance. If block size is increased the bias is lowered, while variance increases. It is true also the opposite, when number of blocks increase variance reduces and simultaneously bias raises.

In this paragraph we are going to study why Peaks Over Threshold (POT) can be an improved strategy while applying Extreme Value Theory. POT is based on modeling the behavior of extreme values above some high threshold. This kind of modeling enable to exploit a larger amount of data. Furthermore, POT offers the benefit of easily computing risk measures such as Value-At-Risk and Expected Shortfall.

"Let $X_1, X_2, \dots, X_n$ be a sequence of i.i.d. random variables representing risks or losses with an unknown CDF, F. For the class of distributions F such that the Cumulative Distribution Function of the standardized value of M_n converges to a GEV distribution, it can be shown that for large enough u, there exists a positive function $\beta(u)$ such that the excess distribution $F_u(z)$:

$$F_u(z) = P(X-u \le z \mid X>u) = \frac{F(z+u)-F(u)}{1-F(u)},\ z>0$$

is approximated by the generalized Pareto distribution (GPD) G.[2]"
This postulate suggests that Excess distribution and cumulative pareto distribution G are equivalent. In particular:

$$G_{\xi,\beta(u)}(z) = \begin{cases} 1-\left(1+\left(\frac{\xi z}{\beta(u)}\right)^{-1/\xi}\right) & \xi \neq 0 \\ 1-\exp\left\{-\frac{z}{\beta(u)}\right\} & \xi = 0 \end{cases}$$

The parameter ξ determines the shape of the distribution:

[2] Viviana Fernandez, extreme value theory: value at risk and returns dependence around the world

- ξ<0: H_ξ is a Pareto type II distribution;
- ξ=0: H_ξ is an Exponential type distribution;
- ξ>0: H_ξ is a Pareto type I distribution.

It is important to note that GEV and GPD shows a close relationship, in fact, ξ represents in both cases the shape parameter. Whereas $\beta(u)$ corresponds to u and σ.

The major issue when dealing with POT is deciding the optimal threshold u. There are several methods which can be chosen to find u. These include a rule of thumb and a graphical method based on the empirical mean excess function. The former method recommends setting u such that P(X> $u) \approx 10\% - 15\%$. The graphical method is built on the empirical mean excess function:

$$\hat{e}(u) = \frac{1}{n_u}\sum_{i=1}^{n_u}(X_{(i)} - u)$$

Where $X_{(i)}$ represents value of X which are above the previously set u and n_u is the number of occurrences. Theoretically, it can be proved that if the GEV approx. works, then $e(u)$ should be linear in u.

By analyzing $\hat{e}(u)$ plot we can establish the optimal u. When $\hat{e}(u)$ is linear we expect the GDP convergence is appropriate, thus choosing the smaller value of $\hat{e}(u)$ from which the empirical mean excess function is linear and increasing as threshold u. The rationale behind this follows the mean excess function $e(u)$ principle, where $e(u)$ is linear in u[3].

In this section we explain the concept of threshold u using Google stock daily prices. Using R programming language, we transformed daily prices into daily returns, which are fundamental into threshold u estimation. Creating an Empirical mean excess plot applying data observed allows to find the optimal threshold u.

[3] Viviana Fernandez, extreme value theory: value at risk and returns dependence around the world

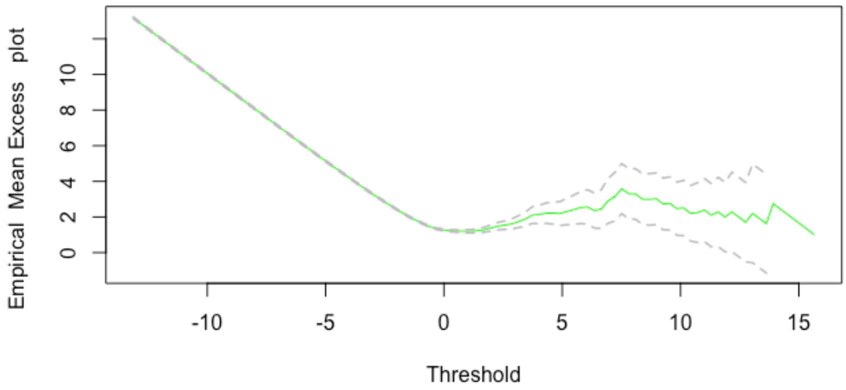

Figure 2 Google mean excess plot (from = "2004/12/31", to = "2022/12/31")

The empirical mean excess plot reported above shows that an optimal value for threshold u could be 2 or 2.5, where the linear shape is evident, and the slope of the curve is increasing.

One of the main differences between Block Maxima method and POT is that the former is affected by inefficiency as it is time-consuming and can lead to extreme values exclusion. Furthermore, POT utilizes more data ending up being a more efficient way of modeling extreme values.
In conclusion, we can state that POT allow for an easier risk measure computation. Value at Risk (VaR) and Expected Shortfall (ES) can be easily computed in the context of POT. As evidence of how simple it is to compute this kind of risk measure, we are going to show and explain them, including an example based on Google returns:

"The Value-at-Risk at level α for the losses, where usually α ranges between 0.95 to 0.99, is: [4]

$$\Pr(X > \mathrm{VaR}_\alpha) = 1 - F(\mathrm{VaR}_\alpha) = 1 - \alpha$$

$$\mathrm{VaR}_\alpha = F^{-1}(\alpha)$$

By setting α and inverting $\hat{F}(x)$, an estimate for the VaR_α can be computed as:

[4] Formulae taken from Professor Raggi course on Risk Management.

$$\widehat{VaR}_\alpha = u + \frac{\hat{\beta}}{\hat{\xi}}\left(\left(\frac{n}{k}(1-\alpha)\right)^{-\hat{\xi}} - 1\right)$$

Here u is the threshold chosen; $\hat{\beta}$ and $\hat{\xi}$ are respectively the scale and shape parameter estimates; n is the number of observations and k the number of losses exceeding the threshold u. The optimal solution to find estimates is by applying maximum likelihood methods to our sample.

It is easy to prove that Expected Shortfall (ES) is defined as the average loss given that VaR_α is exceeded, in case of GPD it can be shown that:

$$\widehat{ES}_\alpha = \frac{\widehat{VaR}_\alpha}{1-\hat{\xi}} + \frac{\hat{\beta} - \hat{\xi}u}{1-\hat{\xi}}$$

In the next lines we show an application of these two formulas on Google return time series. By applying Maximum likelihood estimation to Google data, we obtain the following parameter estimates; based on u=2:

- $\hat{\xi}$ = 0.2335823
- $\hat{\beta}$=1.0690140

The shape parameter estimates $\hat{\xi}$ indicates that Google distribution is affected by heavy tails. Estimating Value at Risk and Expected Shortfall is particularly straightforward, giving that we just need to compute our estimates through the previous formulas.

	90%	95%	99%	99.5%	99.9%
VaR	1.88	2.66	5.05	6.39	10.48
ES	3.23	4.25	7.37	9.12	14.46

Table 1 Estimating Value at Risk and Expected Shortfall (GOOGLE)

The table above, contains estimates for VaR and ES based on different α levels. $\mathrm{VaR}_{0.95}$ = 2.66, means that with 5% probability losses will be higher than 2.66%, while $ES_{0.95}$= 4.25 is the average loss given that $\mathrm{VaR}_{0.95}$ is exceeded. On graphical inspection, $\mathrm{VaR}_{0.95}$ is represented by the first vertical line whereas $ES_{0.95}$ by the second vertical line. As we can notice, the two risk measures are in the right tail of the distribution. In fact, we are analyzing Google negative returns because we want our risk measure expressed in positive values.

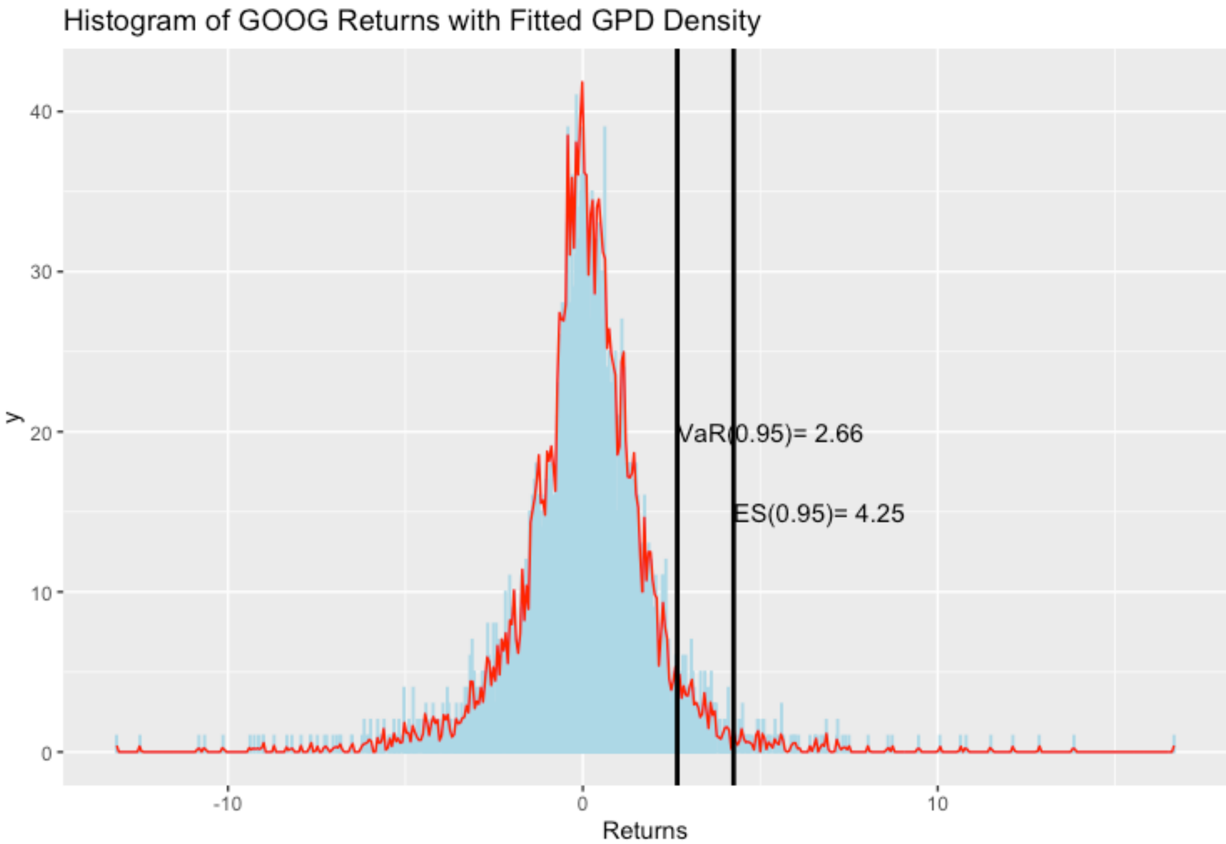

Figure 3 Histogram of GOOGLE returns with fitted GPD density

2. Copula

At the beginning of this thesis, we introduced the idea that classic finance is based on the concept of gaussianity. This assumption has revealed to be not satisfactory while dealing with financial markets due to its features: nonlinear dependence, fat tails and volatility clusters. While in the latter section we showed how Extreme Value theory makes it possible to deal with fat tails, now we show how copulas treat nonlinear dependencies. Copulas offer a useful way to measure the dependence structure between two or more random variables, while considering nonlinear dependence.

It is straightforward to think copula as a distribution between two marginals that described how they are related. The process starts with the marginal distributions of each random variable and after combining them, end up with a copula function. The copula function joins the random variables into a unique multivariate distribution. Choosing marginals is crucial in Copula construction, thus we focus on various kinds of models.

In this chapter we are going to analyze different copula types, starting from their distribution. The element we are focusing on is the possible application of copulas in the context of Extreme Value Theory. Particularly, copulas are graphically represented to provide an intuition on their behavior. We have chosen among the most important copula families and represented them using probability density function and cumulative density function.

Copula is defined as "*a distribution function on* $[0,1]^k$ *with standard uniform marginal distributions, and is denoted as* $C(u_1, u_2, \dots, u_k)$". [5]

As stated before, the aim of a copula is to describe the correlation between marginal distributions. This concept was first studied in Sklar that stated the following result:

"*Let F be a k-variate joint distribution with marginal distributions* $F_1, F_2, \dots, F_k$. *Then there exists a copula C:* $[0,1]^k \rightarrow [0,1]$ *such that for all* $x_1, x_2, \dots, x_k$,

$$F(x_1, x_2, \dots, x_k) = C(F_1(x_1), F_2(x_2), \dots, F_k(x_k))$$

$$= C(u_1, u_2, \dots, u_k)$$

Additionally, if the marginals are continuous then C is unique". [6]

Sklar's theorem is the foundation of copula theory and it explains a modern approach to measure dependence structure. It is important to note that Sklar's theorem has two main implications. On one hand, it demonstrates that copulas can be originated using any known marginal distributions.
On the other hand, as previously stated copula is unique when its margins are continuous.

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[5] Financial Econometrics Notes, Kevin Sheppard University of Oxford, February 2, 2021 p.44

[6] Financial Econometrics Notes, Kevin Sheppard University of Oxford, February 2, 2021 p.44

In conclusion, we can also imagine a Copula as a function that allows us to extract information on the association between the random variables analyzed. In order to do that, we study the marginal distributions and their copula independently, knowing Copula is the function that merge marginal distributions into joint distribution[7].

Copulas families

Copulas are divided in two main families: elliptical and Archimedean copulas. Elliptical Copulas are derived from a standard distribution, among this kind of copulas we have Gaussian and Student's t copula, which are directly derived from the two famous univariate distribution. On the other hand, Archimedean copulas are not based on Sklar's theorem as reported by Stander in "The use of copulas in risk management''. This kind of copulas have a closed form expression and allow to perform an analysis with broader assumptions: such as hypotesing asymmetrical dependence structure.[8] In other words, Archimedean copulas represent a more suitable tool to shape correlation regarding the financial world as in many cases we face asymmetric correlation. For example, it is easier to observe higher correlation value in big losses than correlation value in significant profit.

Gaussian

The Gaussian is one of the most used copulas. It is defined as:

$$C(U,V) = \Phi_\rho(\Phi^{-1}(U), \Phi^{-1}(V))$$

Where $\Phi(\cdot)$ denote the normal cumulative probability function and $\Phi^{-1}(\cdot)$ its inverse. $U, V \in [0,1]$ are uniform random variables and $\Phi_\rho(\cdot)$ the bivariate normal with correlation coefficient ρ.'[9]

This function allows us to merge the two univariate distributions into a single bivariate distribution, making possible analysis on various types of correlation dependence

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[7] Financial risk forecasting, Jon Danielsson p.26

[8] Modelling Dependence with Copulas and Applications to Risk Management, Paul Embrechts, Filip Lindskog and Alexander McNeil

[9] Financial risk forecasting, Jon Danielsson 1.8.1 p.25

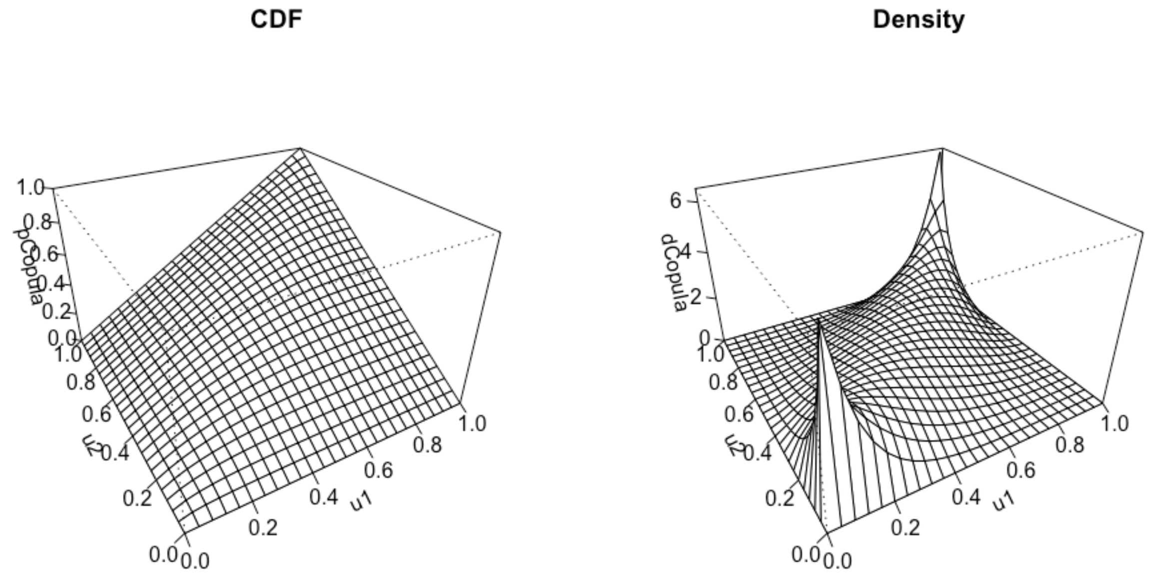

Figure 4 CDF and PDF Normal Copula (ρ=0.8)

The two plots represent the Cumulative distribution function (CDF) and the Probability density function (PDF) of a Normal Copula with correlation coefficient ρ equals to 0.8, indicating a strong positive interdependence between the two marginals. The perspective CDF points out the simultaneous probability of two random variables u_1, u_2 being less than or equal to specific values. Subsequently, by analyzing this graph, it is possible to estimate the probability of various joint events occurring. This property is crucial in risk management since it allows us to assume on specific marginal distributions identifying the correct correlation structure. The PDF's surface expresses the probability density at different points within the unit square, underlying concentration in probability mass. By examining this perspective plot, we get an overview of how density changes across the unit square. For example, peaks in the plot represent area with higher or lower probability to observe determined combinations of u_1, u_2. The PDF perspective plot allows us to understand how much it is probable for a joint event to take place.

In the context of risk management perspective PDF is critical to understand the likelihood of extreme financial market events. In this PDF plot we observe higher value in the initial and final value of the unit square, indicating a high probability of extreme events. Analyzing perspective plot can be challenging due to its 3-dimensional representation and its complexity examination. For this reason, we utilize a contour plot which allows us to display a 3-dimensional graph on a 2-dimension plot.In the next section, we analyze a contour plot of the Cumulative distribution function and of the PDF.

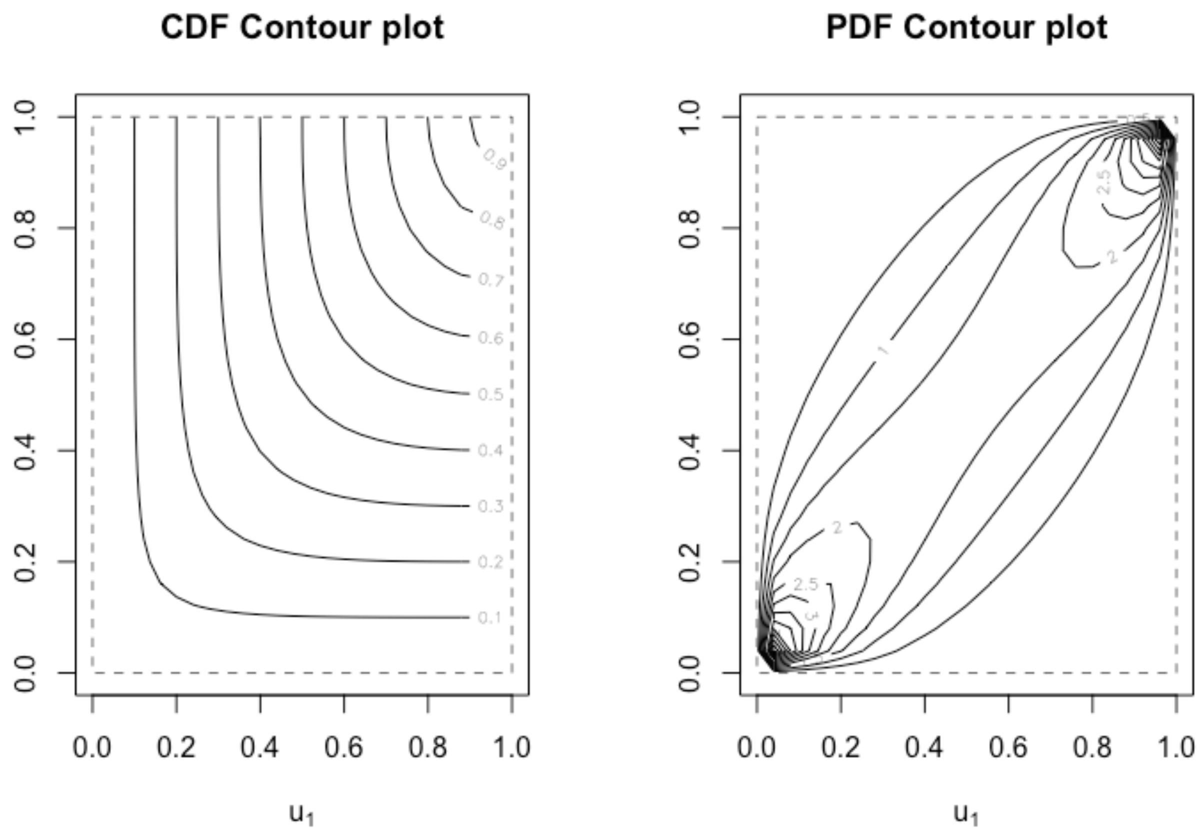

Figure 5 Contours plot for CDF and PDF (Normal Copula, ρ=0.8)

The first graph displays the normal copula’s cumulative distribution function. Contours link points with same cumulative probability in the unit square, easing the visualization of areas with the equal likelihood. To offer a clearer explanation, by following contour lines we can identify regions with similar probabilities of observing determined joint events.

The second plot describes normal copula’s probability density function. Different contours indicate different probabilities, while a single line represents ranks with the same likelihood along the unit square. In line with our previous observation, we can state that PDF contour plot indicates higher extreme event probability in the tail. In fact, contour plot shows darker shades in upper right and in lower left corners. This kind of visualization helps us to comprehend tail heaviness and correlation of various financial assets.

In conclusion, we can say that contour plots are very important tools for understanding features and characteristics of normal copula and other kinds of copulas. In the next paragraphs we are going to show a variety of copulas, both theoretically and graphically, that will be crucial in our analysis.

Student's t

One of the most important elliptical copulas is Student's t copula, which is described by:

$$c(u_1, \dots u_N) = T_{\rho,\nu}\left(t_\nu^{-1}(u_1), \dots, t_\nu^{-1}(u_N)\right)$$

Where $T_{\rho,\nu}$ represents the standardized Student t-distribution with ν degrees of freedom, ρ denotes correlation matrix and $t_\nu^{-1}(.)$ is the inverse of the univariate marginal distribution function.'[10]

Student's t copula is a suitable tool to model symmetric dependence structure characterized by relatively heavy tails, meaning that the two marginal distributions assign high probability on tail events. In financial markets this is a classical scenario, where returns usually do not follow normal distribution. For this reason, student's t copula represents a better solution to shape correlation between random variables and an improvement compared to gaussian's copula capacity to capture heavy tails.

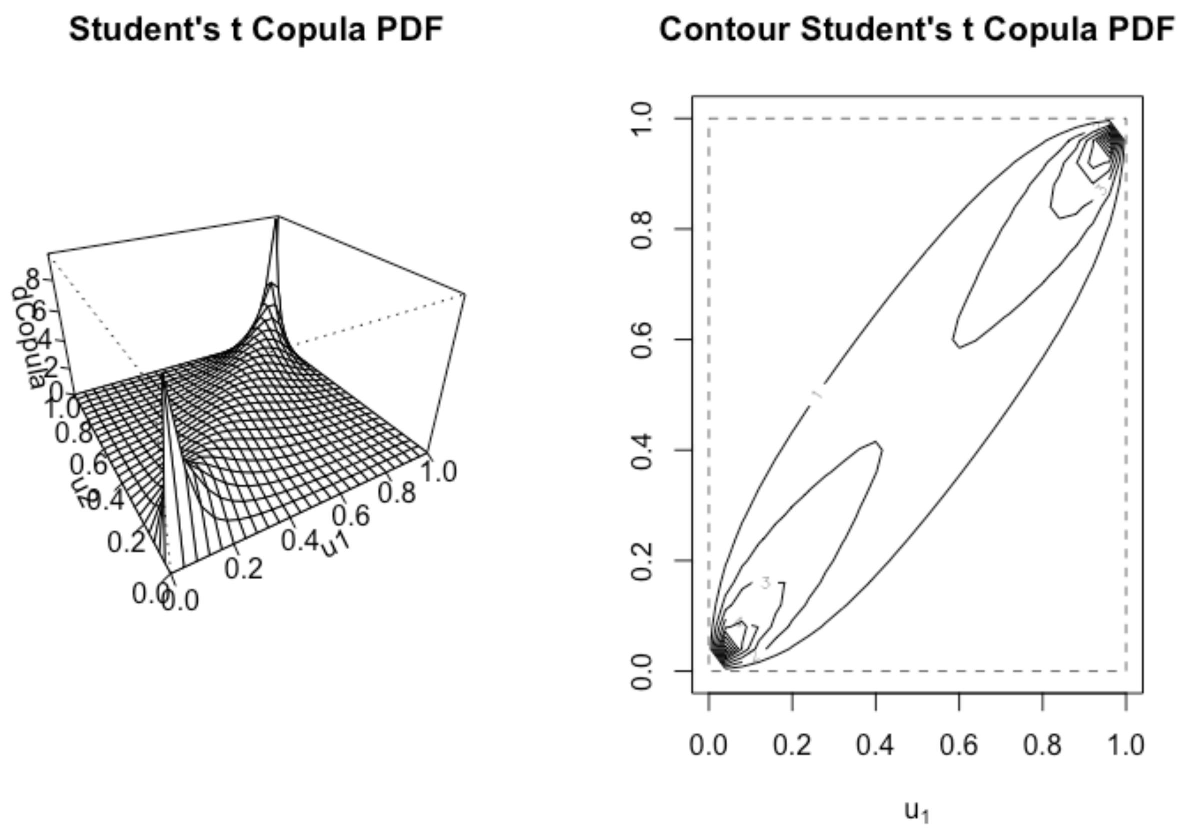

Figure 6 Student's t Copula PDF (ρ=0.8; DoF=3)

[10] The use of copulas in risk management (Stander) p.7

The plot above shows two kinds of PDF representations, already introduced while talking about Gaussian copulas. The Student's t copula has correlation parameter rho=0.8 and 3 degrees of freedom, indicating a high grade of codependence and kurtosis. Student's t copula shows symmetric tail heaviness, influenced by the degrees of freedom. The main difference compared to Gaussian's copula lies in kurtosis specifications, as we can note higher values in the tails for student's t copula.

On the other hand, the plot below describes two kinds of CDF representations for the same student's t copula. Graphically, this is also similar to Gaussian's CDF copula where cumulative probability increases gradually.

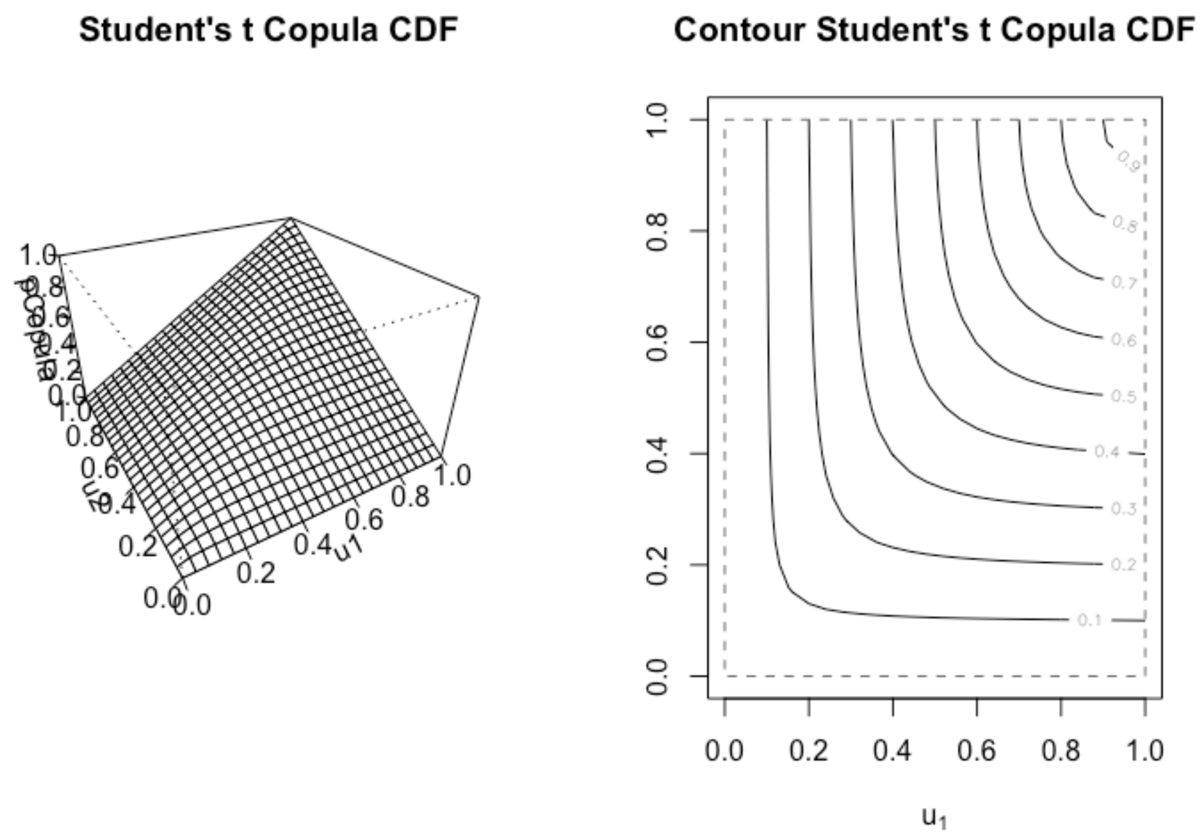

Figure 7 Student's t Copula CDF (ρ=0.8; DoF=3)

Archimedean Copulas

Now, we are going to show one of the most frequently used copula, namely, the Archimedean copula.

Archimedean Clayton Copula

Clayton Copula is described by the following equation:

$$C(u,v) = \left(u^{-\delta} + v^{-\delta} - 1\right)^{-1/\delta}$$

where $0 < \delta < \infty$.[11] This kind of copulas only shows lower tail dependence. Clayton Copula is crucial in describing random variables that show left tail dependence. This is important for returns distribution analysis. For example, considering Google and Apple stocks is more likely that a sharp decrease of Google share price determines a strong decrease in Apple price with higher probability. In other words, simultaneous negative movement are more likely than simultaneous positive movement in stock prices. The plot below shows Clayton Copula PDF with δ= 0.8, highlighting left tail dependence.

[11] Eric Zivot and Jiahui Wang (2005), Modelling Financial Time Series with S-PLUS, Second Edition, p. 729

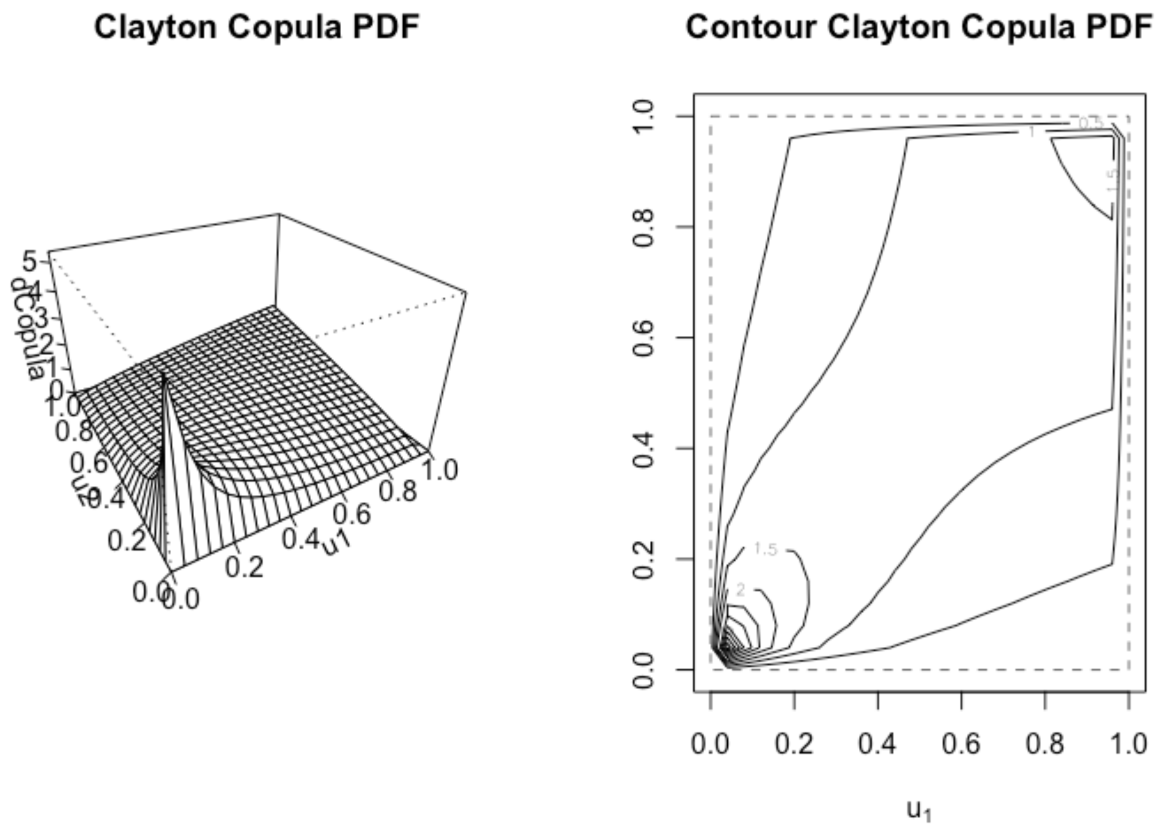

Figure 8 Clayton Copula PDF (δ= 0.8)

Gumbel

As reported in Zivot and Wang (2005) Gumbel Copula is characterized by a dependence parameter δ:

$$C(u,v) = \exp\left\{-\left[\left(-\ln(u)^{\delta} + \left(-\ln(v)^{\delta}\right]^{1/\delta}\right\}, \delta \geq 1$$

δ is defined between 1 and $+\infty$, in case δ=1 is called Independence copula. The higher the value of the dependence parameter, the larger the correlations between the two random variables. Gumbel Copula is particularly famous because it shows upper tail dependence, making it a suitable copula to model Loss distribution or others typical financial phenomena.

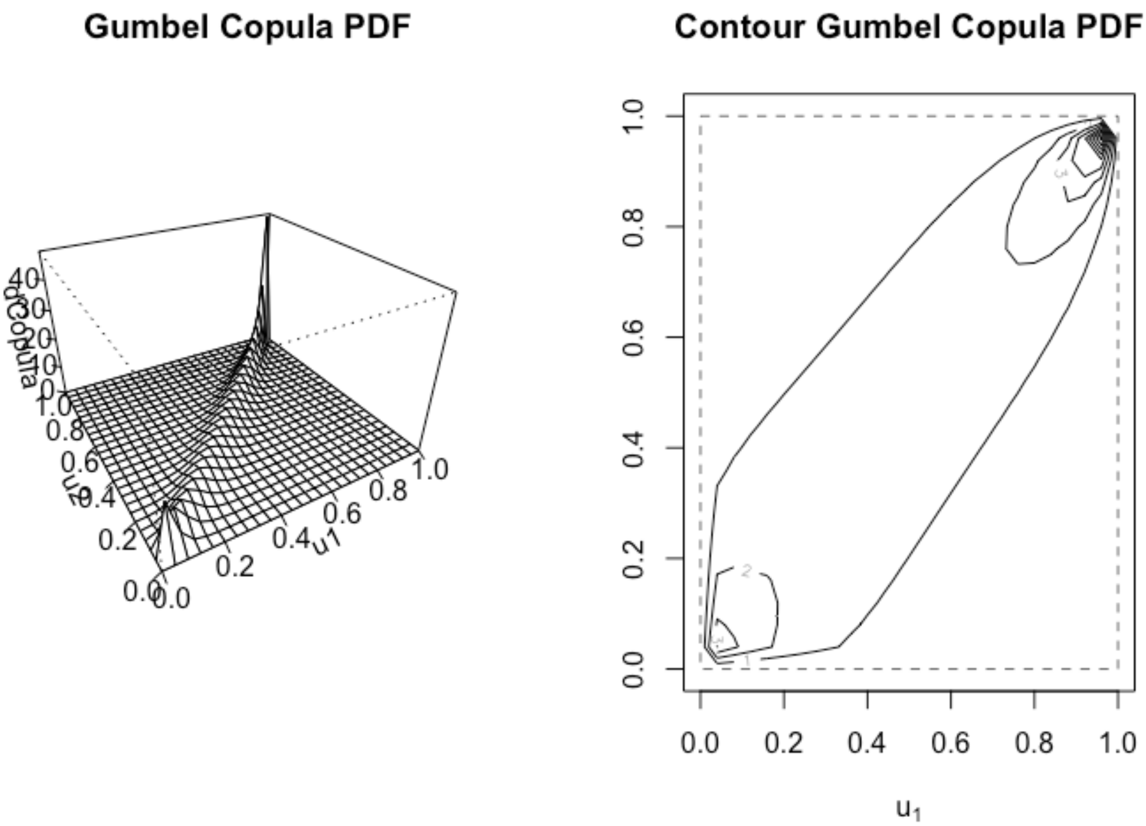

Figure 9 Gumbel Copula PDF (δ=8)

The plot above describes PDF of a Gumbel copula with δ=8. As we can see the right tail of the distribution is heavier than the left tail, indicating tail dependence asymmetry. This feature can be helpful when dealing with Loss distribution or negative returns distribution. In fact, both kinds of distribution are usually right skewed showing a higher probability of extreme events in the upper tail.

Frank

Frank Copula is based on the following copula function[12]:

$$C^{\mathrm{Fr}}(u_1, u_2) = -\frac{1}{\theta}\log\left\{1 + \frac{\left(e^{-\theta u_1} - 1\right)\left(e^{-\theta u_2} - 1\right)}{e^{-\theta} - 1}\right\}$$

[12] D. Ruppert (2011), Copulas, Washington

Where θ is defined in (-∞,+∞) in case θ=0 the Frank Copula is the Independence Copula. Frank Copula is considered a symmetric dependence copula, as elliptical copulas. In the example below, we choose θ=8 as our correlation parameter. The PDF plot exhibits joint distribution of two marginals. The structure is clearly symmetric, as we can observe in both graphs.

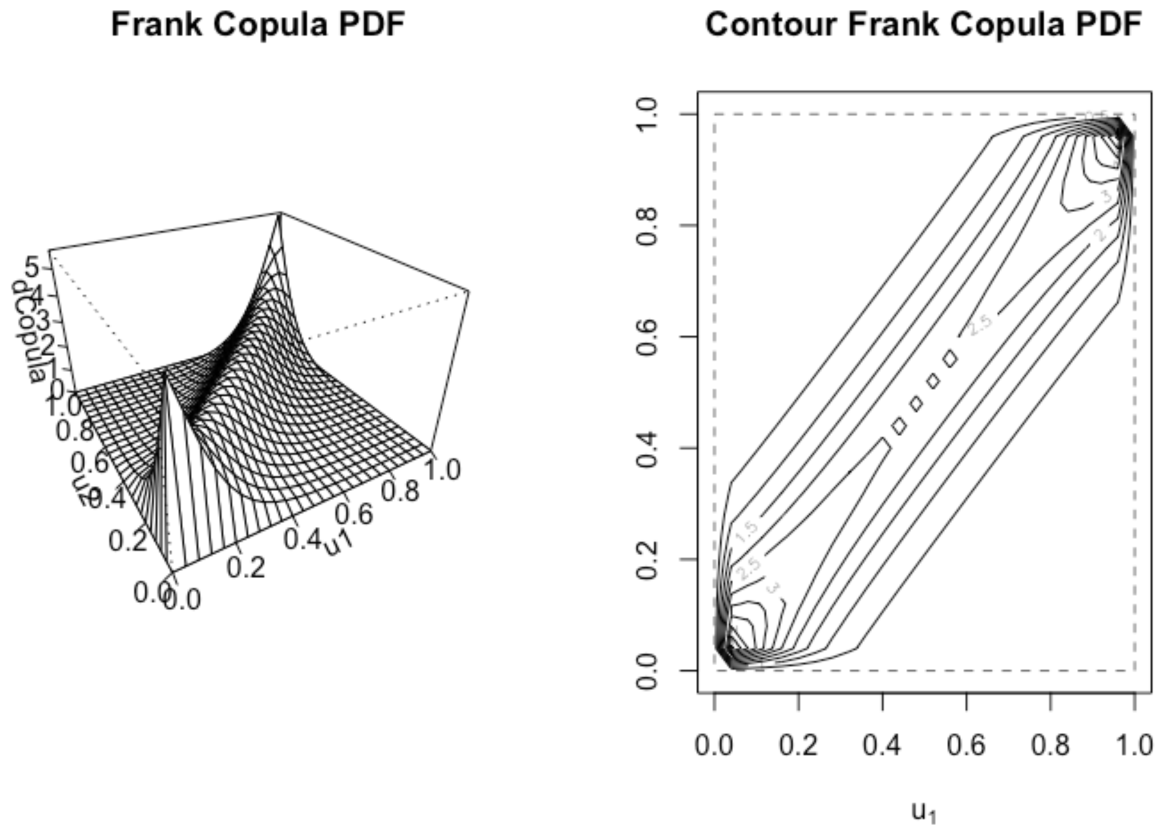

Figure 10 Frank Copula PDF (ϑ=8)

BB1

BB1 Copula is described by the following equation:[13]

$$C(u,v) = \left(1 + \left[\left(u^{-\theta} - 1\right)^{\delta} + \left(v^{-\theta} - 1\right)^{\delta}\right]^{-1/\theta}\right)$$

[13] Modelling Financial Time Series with S-PLUS, Second Edition, Eric Zivot and Jiahui Wang

Where $\theta > 0, \delta \geq 1$ are the copula dependence parameters. BB1 copula shows high modelling flexibility, allowing case with either left or right tail correlation depending on the paramaters chosen.[14] From BB1 copula derives two types of copula we have already encountered:

- When θ=0, BB1 copula is the Clayton Copula;
- When δ=1, BB1 copula is the Gumbel Copula.

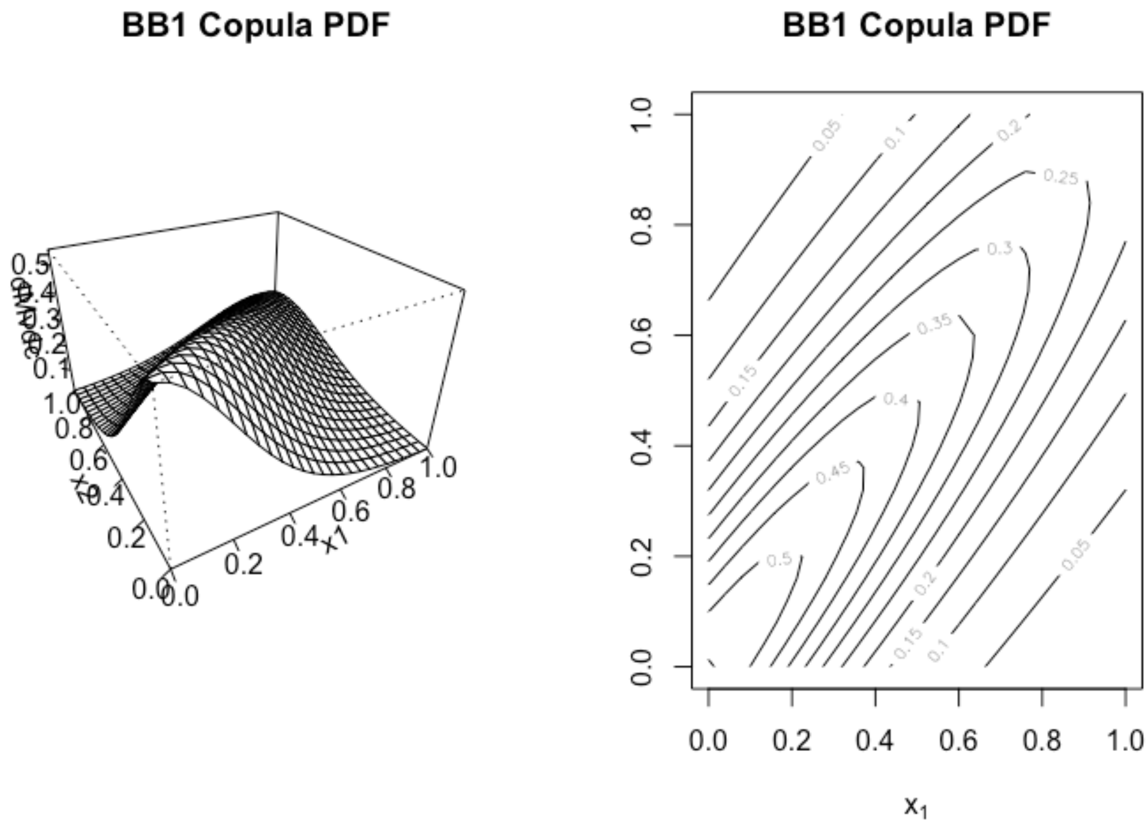

Figure 11 BB1 Copula PDF (ϑ=2, δ=3)

In the plot above is displayed a BB1 copula with θ=2 and δ=3. In this case we constructed BB1 copula starting by two marginal distributions, we chose two student's t with 3 degrees of freedom.

[14] Copula-based top-down approaches in financial risk aggregation, December 2006, Christian Cech

Copula analysis on Google and Apple stocks

In this paragraph we show which kind of copula could be the best fit while analyzing dependence structure between Google and Apple returns. We start by cleaning time series data in R obtaining negative returns for both stocks. After that, we fit different copula models and verify their fit to data through loglikelihood, AIC and BIC.

COPULA FAMILY	LOGLIKELIHOOD	AIC	BIC
GAUSSIAN	812.75	-1623.51	-1617.09
STUDENT'S T	1019.44	-2034.87	-2022.04
CLAYTON	820.74	-1639.48	-1633.06
GUMBEL	816.46	-1630.93	-1624.51
BB1	969.67	-1935.34	-1922.50

Table 2 GOOGLE and APPLE copulas fit

Analyzing this table allow us to choose between various copula families which one represents the best fit for our data. It appears clear from the results that student's t copula is the best fit, it has the higher loglikelihood while having lower values for both AIC and BIC, ensuring the best tradeoff between fit and model complexity. Following the same criteria, we can consider BB1 copula as a good shaping for our data. In the end, we chose as third example Gumbel Copula for its upper tail dependence. Parameter estimates are summarized in the following table:

COPULA FAMILY	1° PARAMETER	2° PARAMETER	KENDALL'S TAU
STUDENT'S T	0.56	3.06	0.38
BB1	0.54	1.29	0.39
GUMBEL	1.57	-	0.36

Table 3 Copulas parameter estimates

Student's t parameters refer respectively to correlation parameter ρ and degree of freedoms. BB1 is defined by θ=0.54 and δ=1.29. On the other hand, Gumbel is defined by just dependence parameter δ=1.57. The three models have similar Kendall's tau, which is a correlation coefficient explained in the next paragraph.

Now, we are going to show graphical representation of the three copula models fitted to the data, using parameter estimates to underline correlation structure between Google and Apple.

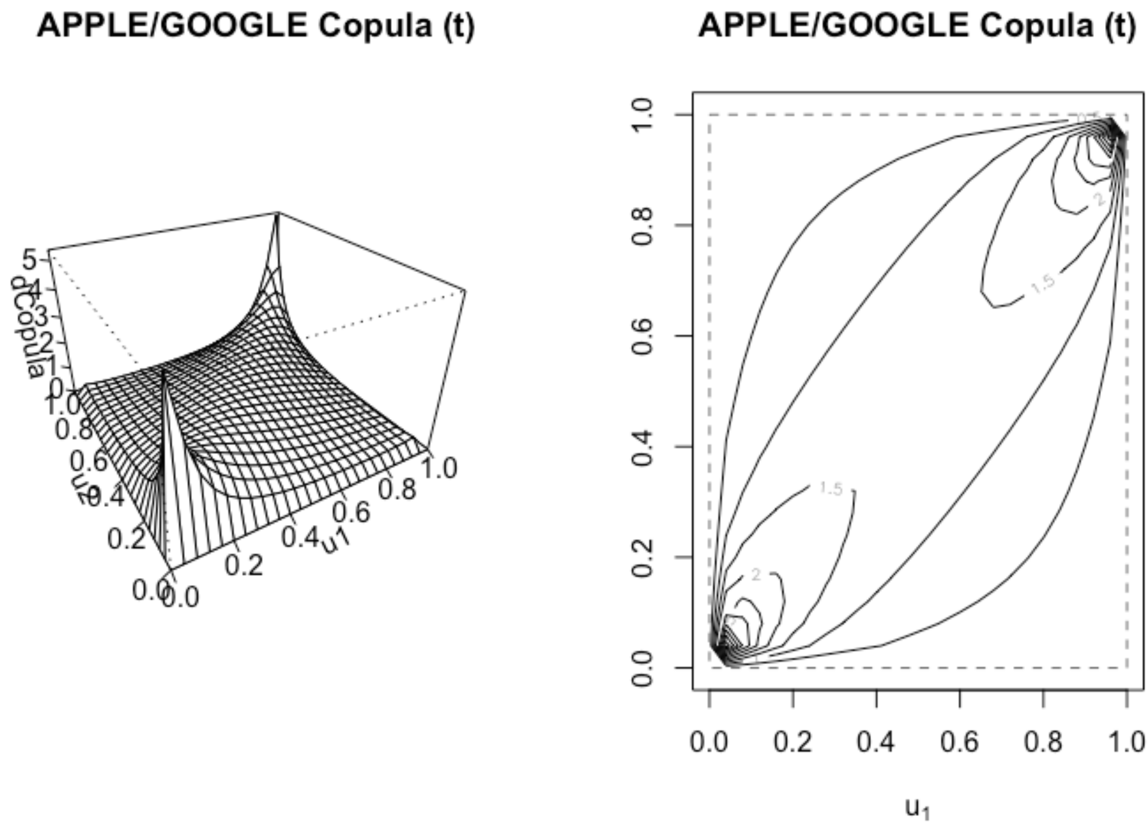

Figure 12 Apple-Google Copula PDF Student's t (ρ=0.56, DoF=3)

The student's t copula PDF suggests that dependence structure between these two stocks can be shaped with a symmetric tail heaviness, meaning that the likelihood of extreme co-movement in the left tail is equal to extreme co-movement in the right tail. For instance, a sharp decrease in Google prices has probability equal to a sharp decrease in Apple price. This is also valid for increase in value, they show the same relationship. Apple/Google copula (t) has the same degree of freedoms of the example in paragraph (copula student t) while a lower correlation parameter rho, ending up having fewer extreme observations in both tails.

In the second model we can observe how left tail is heavier, determining asymmetry dependence. This means that is more probable to observe positive co-movement than observing negative co-movement in the two tails. In this context, this implies that a significant rise in Apple stock price following a considerable increase in Google stock price is more probable than observing two consequential huge losses.

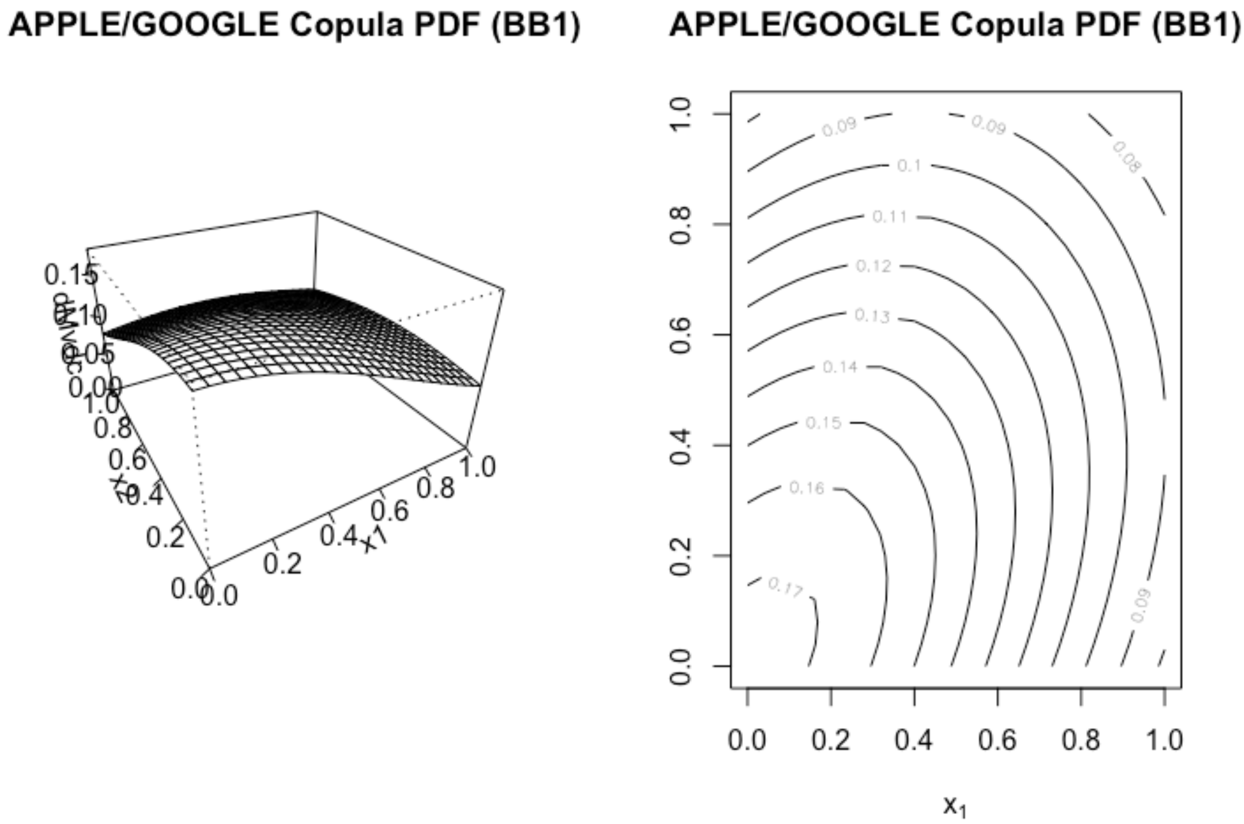

Figure 13 Apple-Google Copula PDF BB1 (ϑ=0.54, δ=1.29)

Gumbel copula is the last model chosen among the possible copula fits. We decide to shape dependence through Gumbel copula due to its tail asymmetry and to show how different copulas can originate various dependence structure. As we can notice in the plot, the right tail is heavier than the left tail. Showing an opposite situation if compared to BB1 fit, in fact, a negative co-movement in the price of both stocks is more probable than a synchronized positive increase in the price of both stocks.

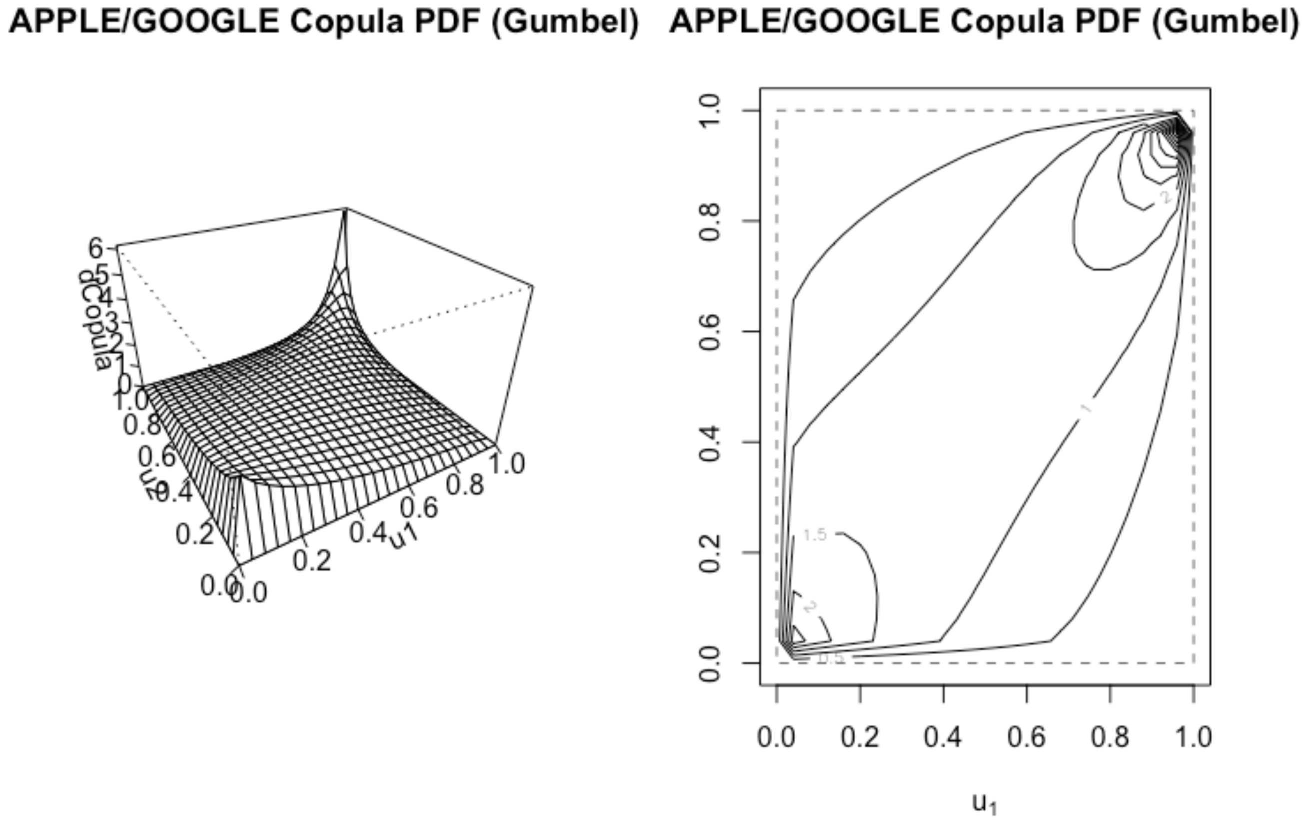

Figure 14 Apple-Google copula PDF Gumbel (δ=1.57)

Tail Dependence

Tail Dependence is a crucial concept when considering copulas. It measures the dependence between two random variables in the tails of the copula distributions, both upper and lower[15]:

$$\lambda_u(X,Y) = \lim_{q\to 1} \Pr\left(Y > \mathrm{VaR}_q(Y) \mid X > \mathrm{VaR}_q(X)\right)$$

$$\lambda_l(X,Y) = \lim_{q\to 0} \Pr\left(Y \leq \mathrm{VaR}_q\,(Y) \mid X \leq \mathrm{VaR}_q\,(X)\right)$$

Applying the first equation we find the probability that Y is larger than a high q-quantile given X is already extreme. On the other hand, this latter formula shows the probability by which Y is lower than a q-quantile of Y given X is below a q-quantile. In the first case q tends to 1, while in the second one tends to zero. If λ_l, λ_u = 0 our copula does not exhibit any tail dependence. An example of copulas that show tail independence are Gaussian and student's t copulas. λ_l, $\lambda_u \neq$ 0 our copula shows tail dependence. As we already discussed before Archimedean Copulas usually are defined as tail dependent copulas. Gumbel copula is an upper tail dependent function, whereas Clayton is a lower tail dependent copula.

Correlation measures

Embrechts, Lindskog and Mcneil (2001) stated that Pearson's linear correlation is a measure of dependence driven by both the univariate distribution and a copula, while Kendall's tau and Spearman's rho rely only on the copula[16].

Embrechts, Lindskog and Mcneil (2001) theorized four essential properties that a universal, single value measure of dependence must follow:

1. $\delta(X,Y) = \delta(Y,X)$
2. $-1 \leq \delta(X,Y) \leq 1$
3. $\delta(X,Y) = 1$ if X and Y are co-monotonic; $\delta(X,Y) = -1$ if X and Y are counter-monotonic
4. If T is strictly monotonic, then:

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15 Modelling Financial Time Series with S-PLUS, Second Edition, Eric Zivot and Jiahui Wang pag. 727

16 Quantitative Risk Management: Concepts, Techniques and Tools, Darrell Duffie and Stephen Schaefer, p.206

$$\delta(T(X),Y) = \begin{cases} \delta(X,Y) & T \text{ increasing} \\ -\delta(X,Y) & T \text{ decreasing} \end{cases}$$ [17]

It is crucial to note that Pearson' s linear correlation satisfies only the two-initial conditions, meaning that Pearson' s linear correlation is considered a symmetric dependence measure that range between -1 and 1. Whereas the two rank correlation measures satisfy all the conditions above. In the next sections we define the three measures of dependence and furtherly explore their features and differences.

Linear correlation is defined as a 'measure of linear dependence:

Let $(X,Y)^T$ be a vector of random variables with nonzero finite variances. The linear correlation coefficient for $(X,Y)^T$ is:

$$\rho(X,Y) = \frac{\mathrm{Cov}\,(X,Y)}{\sqrt{\mathrm{Var}\,(X)}\sqrt{\mathrm{Var}\,(Y)}}$$

where $\mathrm{Cov}\,(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$ is the covariance of $(X,Y)^T$, and $\mathrm{Var}\,(X)$ and $\mathrm{Var}\,(Y)$ are the variances of X and Y.'[18]

Pearson's correlation is defined as perfect positively dependent when its value is +1, perfect negatively dependent when is -1. Moreover, Pearson's correlation is considered null when it equals 0. The last case becomes particularly evident when two random variables are independent.
In the financial world, two assets that are perfectly positively dependent move together by the same amount. Obviously, this is a truly low probability event if not deemed impossible. Pearson's correlation is the most widely known measure of correlation, but we should bear in mind that might not be the most appropriate method, especially when analyzing non-elliptical distributions.
As stated before, Pearson linear correlation counts among its properties only two of the four required to be a proper measure of dependence. The absence of these properties in Pearson's linear correlation underlines the interest on other estimation method for correlation.
In this paragraph we define two measures of dependence that are directly related with the concept of copula function and are considered measure of concordance[19].

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[17] Modelling Financial Time Series with S-PLUS, Second Edition, Eric Zivot and Jiahui Wang, pag. 724

[18] Modelling Dependence with Copulas and Applications to Risk Management, Paul Embrechts, Filip Lindskog and Alexander McNeil pag. 9/10

[19] Concordance: A pair (u_i, v_i) and (u_j, v_j) of the sample is called concordant if either $u_i < u_j$ and $v_i < v_j$ or $u_i > u_j$ and $v_i > v_j$. It is called discordant if either $u_i < u_j$ and $v_i > v_j$ or $u_i > u_j$ and $v_i < v_j$. Taken from chapter 9 p.725

Kendall's tau is defined as:

$$\tau(X,Y) = \mathbb{P}\{(X-\tilde{X})(Y-\tilde{Y}) > 0\} - \mathbb{P}\{(X-\tilde{X})(Y-\tilde{Y}) < 0\},$$

where $(\tilde{X},\tilde{Y})^T$ is an independent copy of $(X,Y)^T$. Hence Kendall's tau for $(X,Y)^T$ is simply the probability of concordance minus the probability of discordance. Let $(X,Y)^T$ be a vector of continuous random variables with copula C. Then Kendall's tau for $(X,Y)^T$ is given by

$$\tau(X,Y) = Q(C,C) = 4\iint_{[0,1]^2} C(u,v)\mathrm{d}C(u,v) - 1.$$

Note that the integral above is the expected value of the random variable $C(U,V)$, where $U,V \sim U(0,1)$ with joint distribution function C, i.e. $\tau(X,Y) = 4\mathbb{E}(C(U,V)) - 1$.'[20]

In this thesis we are interested in estimating empirically τ using an estimator:

$$\hat{\tau} = \frac{1}{\binom{n}{2}} \sum_{1\le i\le j\le n} \mathrm{sign}\left((x_i - x_j)(y_i - y_j)\right)$$ [21]

$\hat{\tau}$ is computed as the difference between concordant pairs and discordant pairs divided by $\frac{n}{2}$. Sample's concordant pairs are simply ranked observations of the two random variables that move together, the ranked observation of one change in the same direction of the other. The mechanism behind discordant pairs is the opposite, we want to know when ranked observation do not move in opposite direction.

Spearman's rho is defined as:

''Let $(X,Y)^T$ be a vector of continuous random variables with copula C. Then Spearman's rho for $(X,Y)^T$ is given by

$$\rho_S(X,Y) = 3Q(C,\Pi) = 12\iint_{[0,1]^2} uv\,\mathrm{d}C(u,v) - 3 = 12\iint_{[0,1]^2} C(u,v)\mathrm{d}u\,\mathrm{d}v - 3.$$

Hence, if $X \sim F$ and $Y \sim G$, and we let $U = F(X)$ and $V = G(Y)$, then

[20] https://people.math.ethz.ch/~embrecht/ftp/copchapter.pdf pag.13

[21] Modelling Financial Time Series with S-PLUS, Second Edition, Eric Zivot and Jiahui Wang

$$\begin{aligned}\rho_S(X,Y) &= 12\iint_{[0,1]^2} uv\,\mathrm{d}C(u,v) - 3 = 12\mathbb{E}(UV) - 3\\ &= \frac{\mathbb{E}(UV) - 1/4}{1/12} = \frac{\mathrm{Cov}\,(U,V)}{\sqrt{\mathrm{Var}\,(U)}\sqrt{\mathrm{Var}\,(V)}}\\ &= \rho\big(F(X),G(Y)\big).\end{aligned}$$ [22]

The estimator for $\rho_S(X,Y)$ is: [23]

$$\hat{\rho}_S = \frac{12}{n(n^2-1)}\sum_{i=1}^{n}\left(\mathrm{rank}\,(x_i) - \frac{n+1}{2}\right)\left(\mathrm{rank}\,(y_i) - \frac{n+1}{2}\right)$$

The underlying concept of this formula is to exploit squared rank differences to derive the estimator.
As mentioned above, the fundamental difference between rank correlation and linear correlation rely on the capacity of the former to detect both co-monotonicity and counter-monotonicity.

[22] Modelling Financial Time Series with S-PLUS, Second Edition, Eric Zivot and Jiahui Wang

[23] Modelling Financial Time Series with S-PLUS, Second Edition, Eric Zivot and Jiahui Wang, p.726

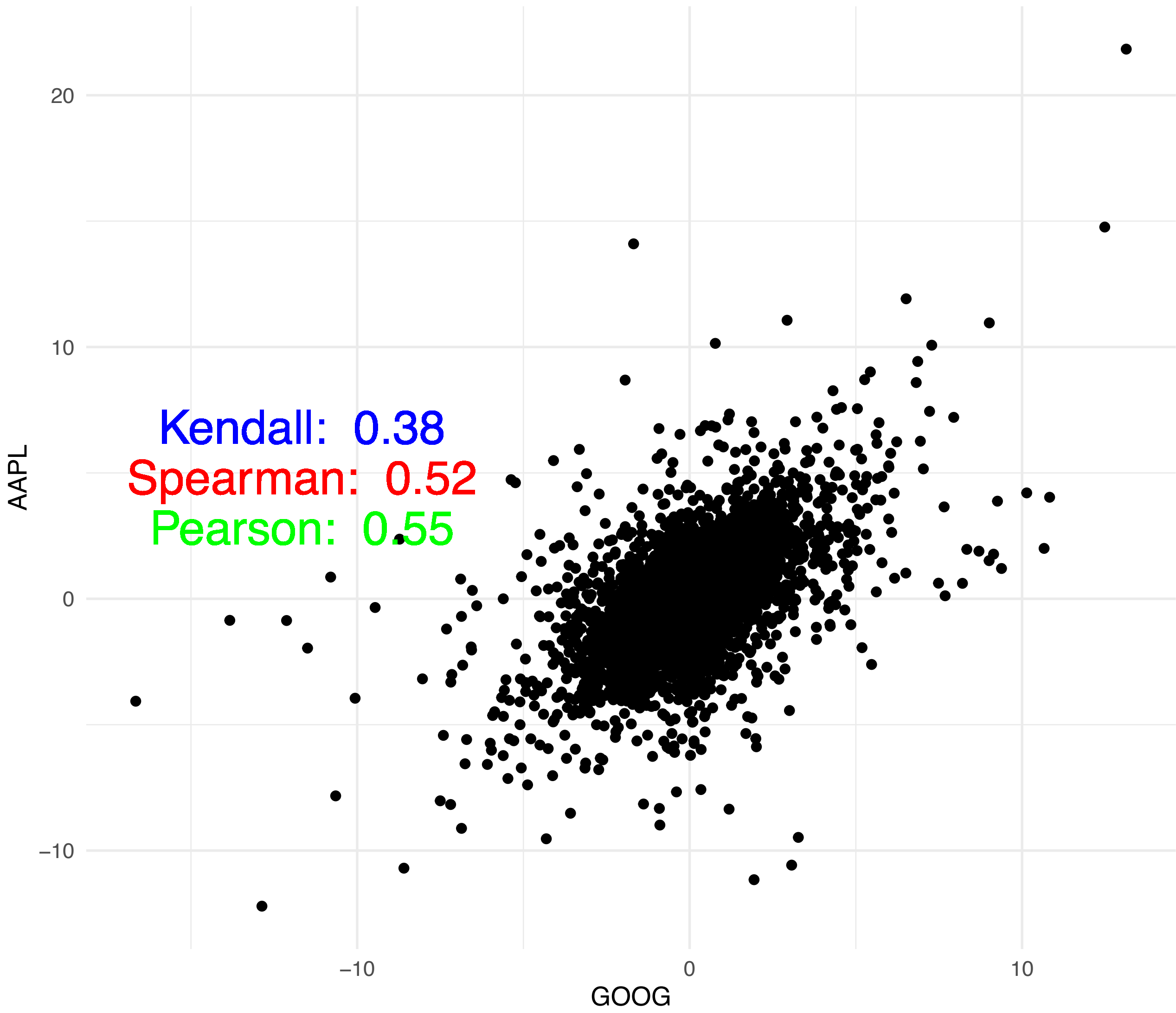

Figure 15 Correlation Scatterplot and correlation measures

In this plot we analyze returns of Apple (AAPL), Google (GOOG) and three different measures of dependence. As we can notice by observing this scatterplot, all the correlation measures detect positive dependence between Apple and Google, while Kendall's tau assumes a lower value than both Spearman's rho and Pearson's rho. Pearson's correlation, shown in green, quantifies the linear correlation between Apple and Google as 0.55. On the same level is Spearman's correlation value (0.52), displayed in red. In the end, Kendall's correlation, shown in blue, has a value equal to 0.38 which implicates a lower correlation compared to the other rank correlation and linear correlation.

Overall, the rationale behind utilizing rank correlation in the context of financial returns dependence is profoundly based on the rank correlation ability to deal with skewed data and nonlinear dependence. As a matter of fact, Financial Markets often exhibit non-normal returns and nonlinear correlation challenging Pearson's correlation as a method of analysis in this context.

Generally financial products show irregular correlation meaning that they co-move with different magnitude. Obviously, nonlinearity increases as the grade of complexity of the product rises.

As previously stated, normality assumptions, is a fundamental property of linear correlation. However, normality assumption is rarely respected by market returns, preferring other skewed or fat-tails distributions.

Copula represents an intuitive and flexible way to shape interdependence between financial returns. Rank correlation plays a significant role in copula modelling, allowing to choose measure of dependence suited to the complexity of nonlinear and nonnormal scenarios. To sum up, rank correlation and copula are a better solution in analyzing and interpreting financial risk, reducing the potential underestimation of risk.

3. Stock Index Analysis: CAC 40 and FTSE MIB

In this chapter we are going to analyze two European main stock indexes: CAC 40 and FTSE MIB. These two stock indexes represent respectively French and Italian top 40 listed companies. Our analysis is based on testing normality assumption, evaluating heavy tailed models and using EVT along with Copula application. Our primary goal is to obtain an optimal fit for our stock indexes to compute risk measures and finding marginal distributions to create our copula. Starting by verifying if the normal assumption is suitable for our data, we are going to fit student's t to our stock indexes and then applying EVT through POT and BM methods. In the last paragraph we will observe a Copula analysis. By selecting marginals obtained by optimal fit and creating optimal copula using maximum likelihood estimation, we aim to show how CAC 40 and FTSE MIB are related to each other.

Analysis on Normality Assumptions

We start by cleaning data and obtaining a bivariate time series on returns. The time series covers the period from December 31st, 2004 to December 31st, 2022. In the first plot CAC 40 returns are displayed, while in the second FTSE MIB returns. At first glance we can observe how the two returns are correlated in times, showing the same direction movements with slightly different peaks.

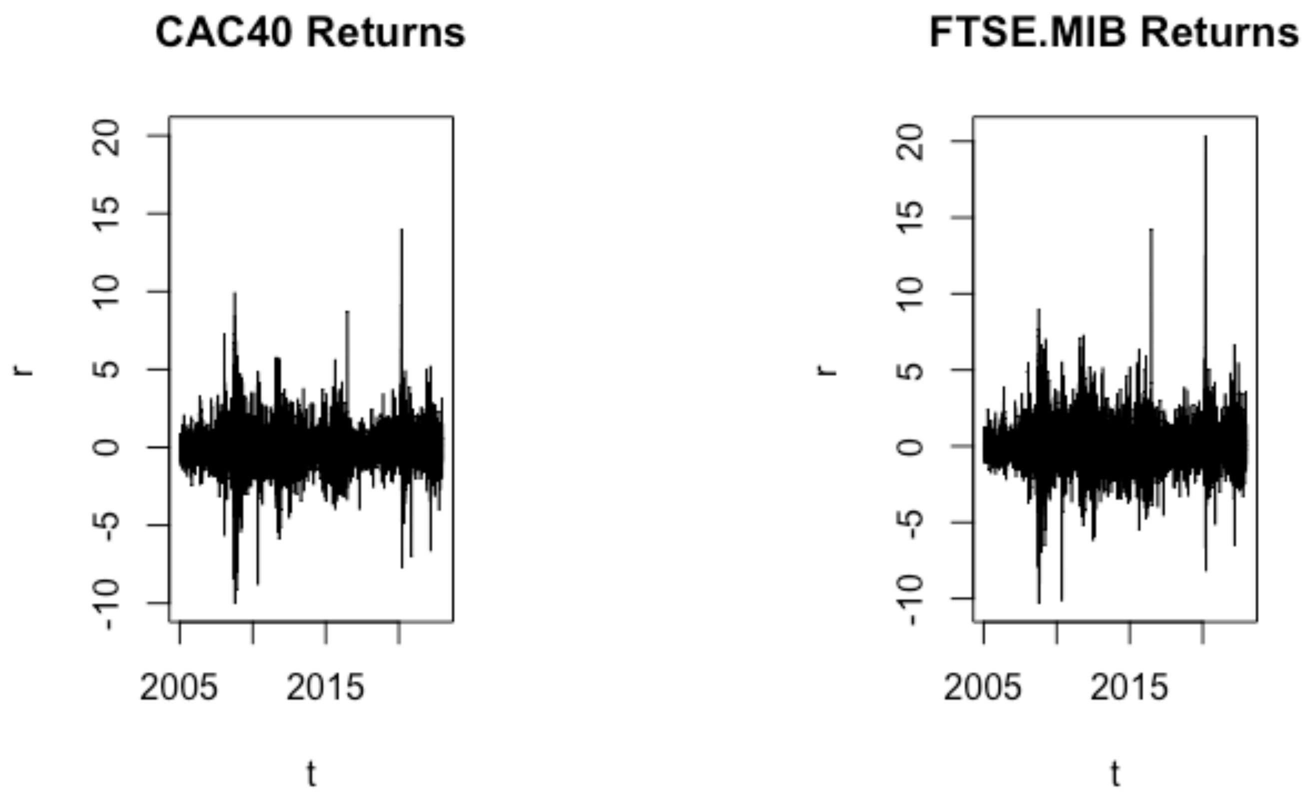

Figure 16 CAC 40 and FTSE MIB returns time series

Graphically, we can notice how the returns are clustered in certain periods. These periods coincide with recent crisis. If we display time series on the same plot we can be even more accurate to describe this pattern.

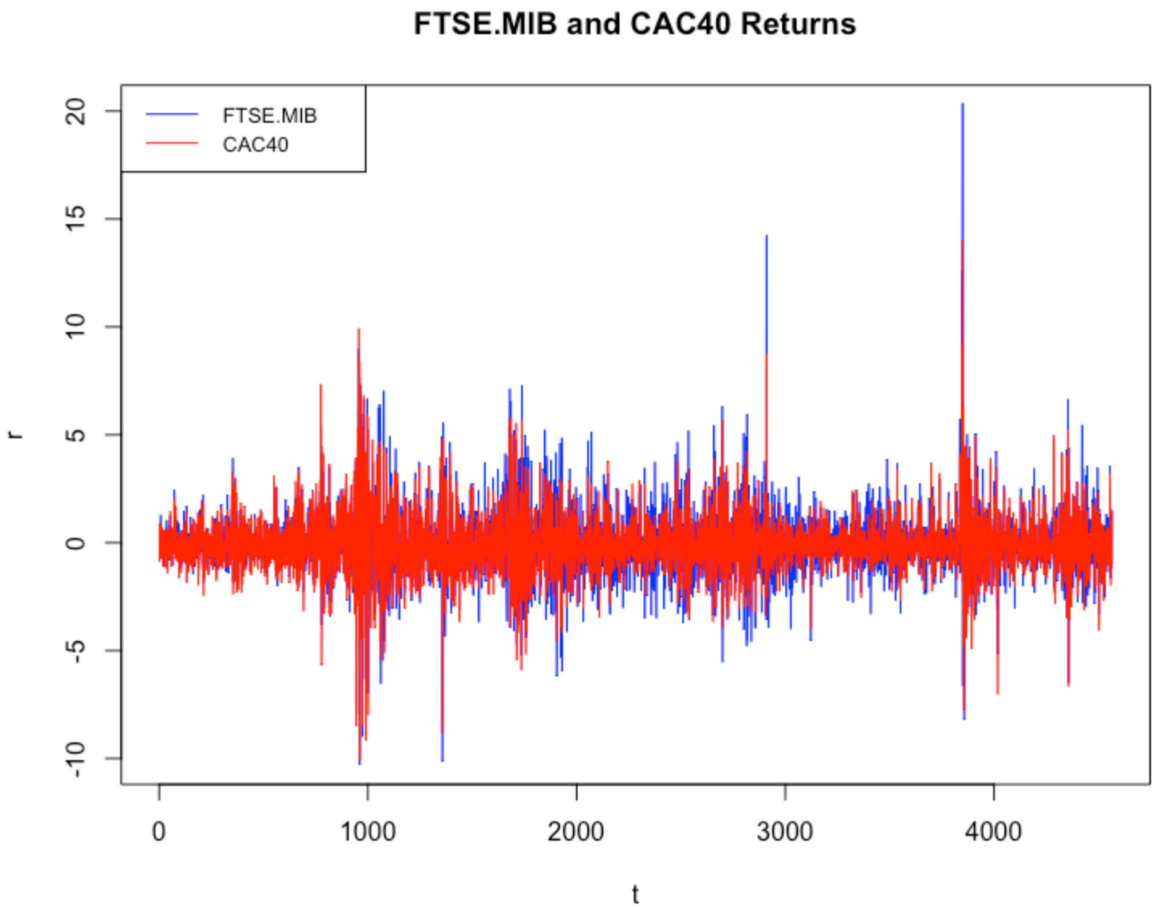

Figure 17 CAC 40 and FTSE MIB returns time series

By including in the same plot the two series, we highlight their interdependency. CAC 40 and FTSE MIB seem to be correlated even on high volatility movements. Now, we want to be more informative about which distributions CAC 40 and FTSE MIB returns follow and particularly if they can be considered Gaussian. In this first part of our dissertation, we are going to discuss normality assumption by considering basic statistics as mean, standard deviation, skewness and kurtosis. In the table below we summarize the results obtained:

	CAC 40	FTSE.MIB
MEAN	0.002672	-0.01811
STANDARD DEVIATION	1.371719	1.561517
SKEWNESS	0.4798222	0.9571906
KURTOSIS	8.651679	12.04192

Table 4 CAC 40 and FTSE MIB statistics

By looking at these statistics we can determine if the two returns time series are distributed as a Normal distribution. Mean is close to 0 for both stocks suggesting that returns are, on average, null. CAC 40 standard deviation is 1.37 while FTSE MIB is even higher, meaning that FTSE MIB returns are more volatile. For a normal distribution skewness is equal to 0 and indicates returns symmetry. In our analysis both FTSE MIB and CAC 40 appear to be asymmetric. In fact, CAC 40 skewness is equal to 0.48 and FTSE MIB to 0.96 suggesting that the two returns distribution are characterized by positive asymmetry. This feature is also demonstrated by minimum and maximum value observed in the two distributions. In the table n.4 we can notice that both right tails are longer than respective left tail. This condition implies that positive returns are more likelihood compared to losses. In the end we analyze Kurtosis, which refers to the amount of probability concentrated in center and tails distribution rather than in the shoulders. Kurtosis in normal distribution is equal to 3, which means that the distribution is not characterized by heavy tails. Other distributions, like Student's t, are characterized by high kurtosis which indicates more probability for tail event. Our results significantly exceed 3, therefore implying both returns time series are characterized by fat tails.
In the end if we analyze CAC 40 and FTSE MIB extreme values, it becomes evident that we expect to observe more positive extreme returns than extreme negative losses.

	CAC 40	FTSE.MIB
MIN	-10.052669	-10.30386
1ST QUARTILE	-0.674465	-0.77715
3RD QUARTILE	0.606872	0.72322
MAX	13.994896	20.37733

Table 5 CAC 40 and FTSE MIB statistics

Given all these signals it becomes evident that normality assumption is not a suitable fit for our data. To solidify this assumption, we introduce normality statistical tests and other graphical visualization such as Q-Q plot and histogram.

In the two tables below, we summarized statistic values for each test. We choose four different statical tests, each of them tests normality assumption. Anderson-Darling and Kolmogorov-Smirnov tests verify if CAC 40 and FTSE MIB data are drawn from a normal distribution while Jarque-Bera test points out if there any fat tails in the distribution.

TEST	VALUE	P-VALUE
ANDERSON-DARLING	68.975	< 2.2e-16
KOLMOGOROV-SMIRNOV	0.56604	< 2.2e-16
JARQUE- BERA TEST	14377	< 2.2e-16
LJUNG-BOX	3.7216	0.05371

Table 6 CAC 40 Normality tests

As reported above, the results clearly show how normality distribution does not fit our data. In fact, each test rejects the null hypothesis which clearly refers to gaussianity assumption. Anderson-Darling test value is 68,975 and its p-value is less than 2.2e-16 strongly indicating a departure from normality. Kolmogorov-Smirnov test value is above critical value whereas its p-value is almost null. Also, Jarque-Bera test takes value that allows to reject normal hypothesis. By analyzing these values, we can state that CAC 40 distribution is not normal.

TEST	VALUE	P-VALUE
ANDERSON-DARLING	62.121	< 2.2e-16
KOLMOGOROV-SMIRNOV	0.55381	< 2.2e-16
JARQUE-BERA	28212	< 2.2e-16
LJUNG-BOX	4.6634	0.03081

Table 7 FTSE MIB Normality tests

Moving to the second table, our analysis extends to FTSE MIB data. Statistic values follows CAC 40 results, confirming our conviction on non-normal nature of FTSE MIB data.

Even if both CAC 40 and FTSE MIB show departure from normality we want to explore this feature graphically. In the next lines, we represent the two stocks by plotting their histograms and Q-Q Plots. These two tools allow us to visualize how returns from each stock are distributed and how these impact on tails distribution.

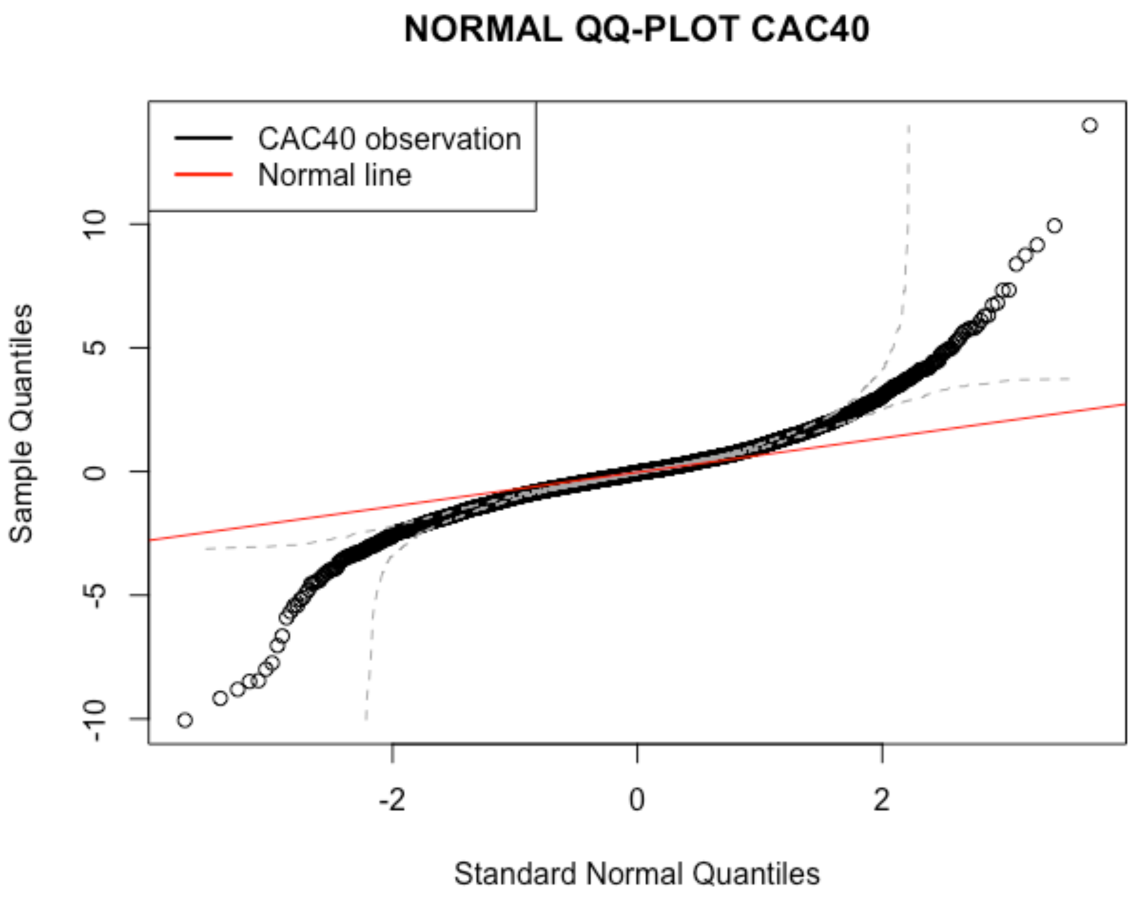

Figure 18 Normal QQ-plot CAC 40

The plot above displays a Q-Q Plot which compares normal quantiles against quantile of CAC 40 returns. At first glance we can notice how many returns observations deviates from normality. In fact, CAC 40 returns lied far away from the Normal line forming a S-shaped matching. So, now we have another tool that suggests departure from normality for CAC 40 returns. In fact, it seems clear that normal distribution is not a suitable fit for our sample data.

We have also another graphical tool that will allow us to firmly exclude gaussianity assumption for CAC 40 data. Histogram helps to visualize returns frequency distributions while the overlay of a normal curve aids to assess if returns follow a normal distribution.

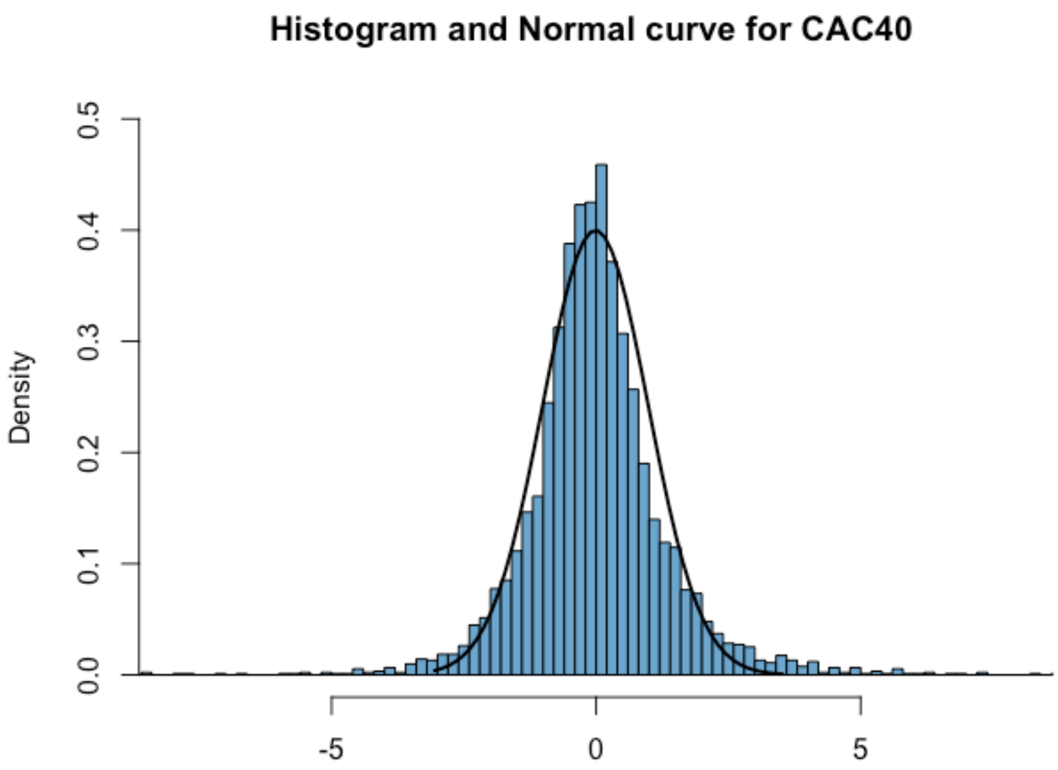

Figure 19 Histogram and Normal curve for CAC 40

Once again, the CAC 40 histogram describes a situation where there is a strong deviation from normality. Bars plot in the tails overcome systematically normal curve. It is important to note that also the central part of CAC 40 distribution is not well described by normal distribution, as evidenced by the bars surpassing normal curve in the plot's middle section.

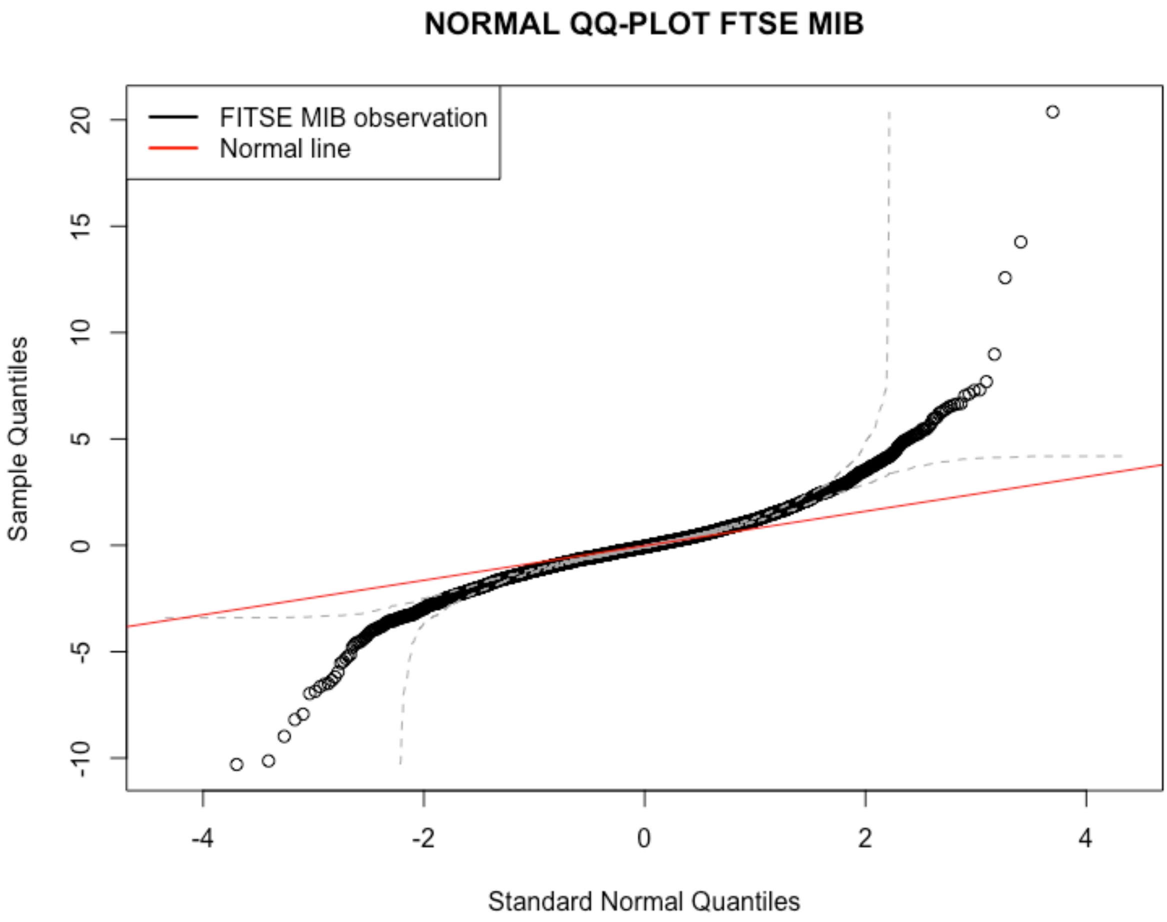

Figure 20 Normal QQ-plot FTSE MIB

The plot above displays a Q-Q Plot which compares normal quantiles against quantile of FTSE MIB returns. We can notice the same dynamic as in CAC 40 with a strong departure from normality. In fact, also CAC 40 returns lied far away from Normal line forming a structure S-shaped. We can conclude that normal distribution is not a suitable fit for FTSE MIB returns.

Now, we are going to see how FTSE MIB histogram fit into Gaussian curve, knowing that likely histogram bars overcome normal curve.

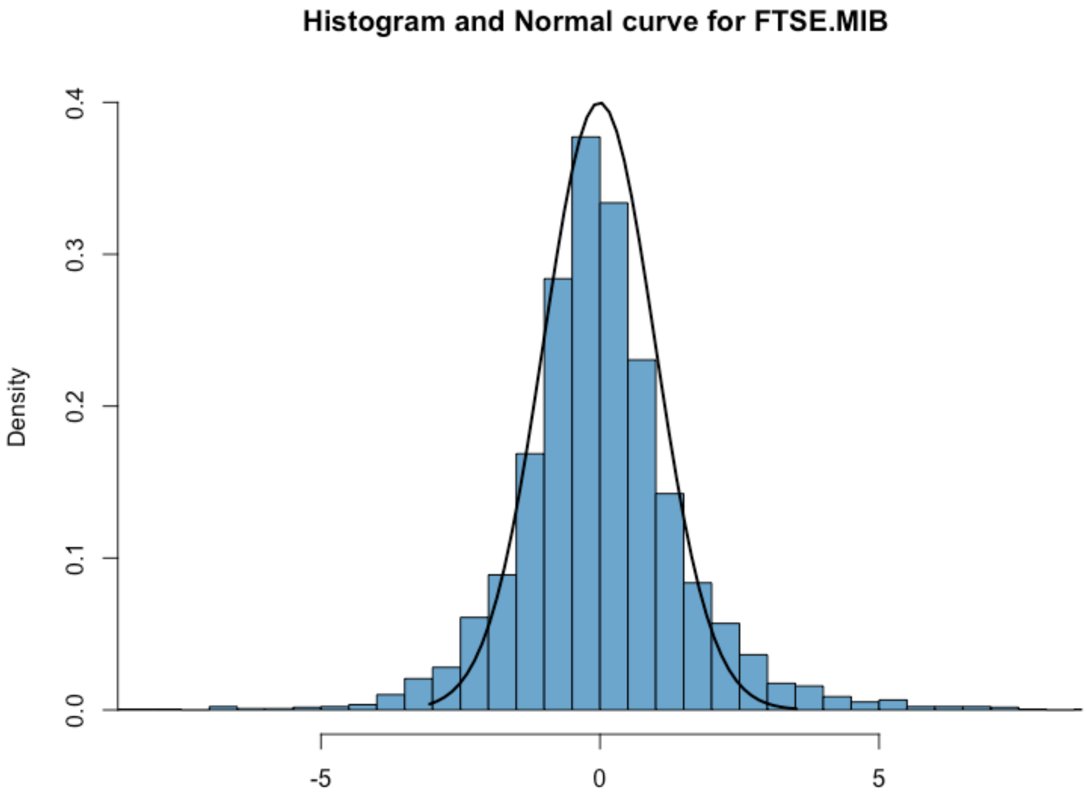

Figure 21 Histogram and Normal curve for FTSE MIB

As anticipated before we can notice how histogram bars represent an almost perfect fit for the central part of the distribution, while normal curve is systemically surpassed in the tails.

In the end, normality assumptions on our returns distributions cannot be considered a credible fit. We should introduce another kind of distribution that allows to better describe our returns, which are characterized by fatter tails than normal distribution.

In other words, our aim is to find a model distribution that could better describe sample data characterized by extreme values, and thus, fatter tails. We will use student-t as a reference distribution and trying to find the best fit using R language. Once we have found the best student-t fit we will show how it represents each of our return distribution.

As reported before, using R language we are going to find the best possible student t in terms of fit for FTSE MIB and CAC 40 by using *"fitdist"* function. *Fitdist* estimates parameters that best match our return distribution. The outcome of R analysis is summarized in the table below:

	CAC 40	FTSE MIB
MEAN	-0.05515526	-0.05302316
STANDARD DEVIATION	1.47670674	1.64009850
DEGREE OF FREEDOM	3.00482283	3.20365077

Table 8 Student's t estimates for CAC 40 and FTSE MIB

Fitdist indicates that the degree of freedom (DoF) for CAC 40 is around 3, while FTSE MIB DoF estimate is slightly higher. The same trend can be observed for standard deviation, where FTSE MIB estimates is higher than the one computed for CAC 40. Mean estimates are nearly identical.

It is interesting to analyze the graphical representation of student-t fitting model and see if tails are well described by this kind of distribution.

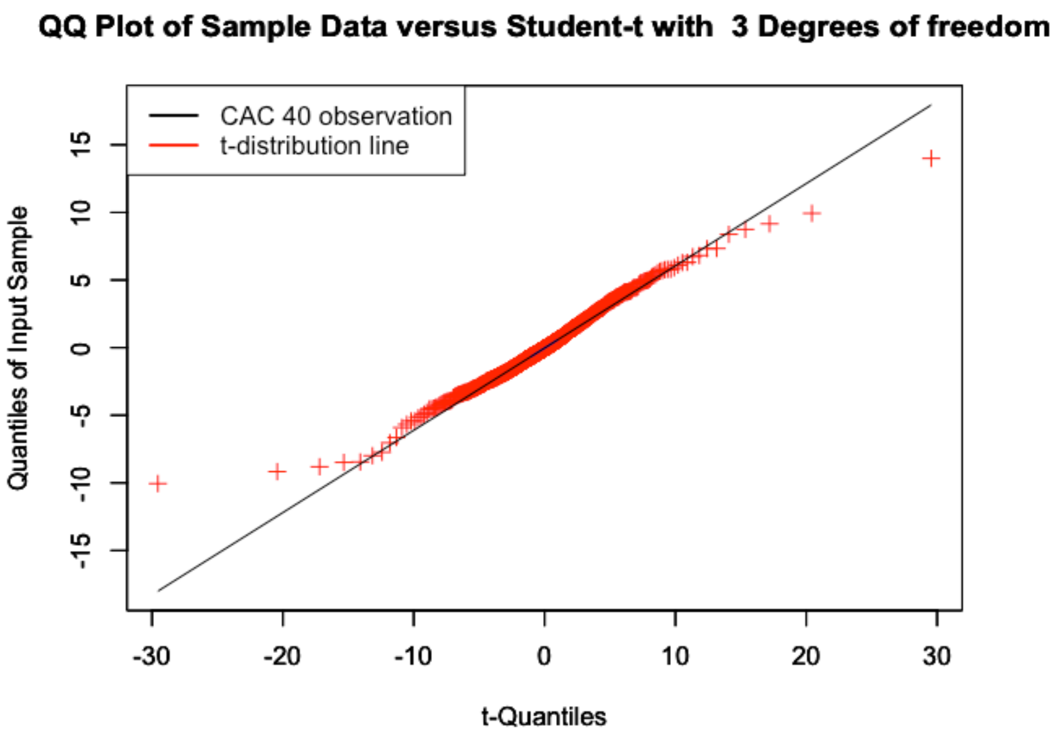

Figure 22 Student's t QQ-plot for CAC 40

Q-Q plot of CAC 40 against Student t with 3 degree of freedom is displayed above. We can appreciate how accurately described are CAC 40 tails. In fact, CAC 40 returns are aligned with student's t distribution. Furthermore, if we compare this Q-Q plot with Normal Q-Q Plot CAC 40 (FIG.) we can note how the two shapes differ. The former appears to be linear while the latter is S-shaped.

The better fit is also highlighted in the next graph. Returns frequency distributions never overcome Student's t with 3 DoF in both extremities. On the other hand, histogram bars in the central part of the plot systematically surpass student's t line indicating sub-optimal fit in the central part. It is interesting to note that the normal line has opposite fitting pattern compared to the student-t line. Given poor results in describing the central part of the distribution we should consider using student-t to assess tail events.

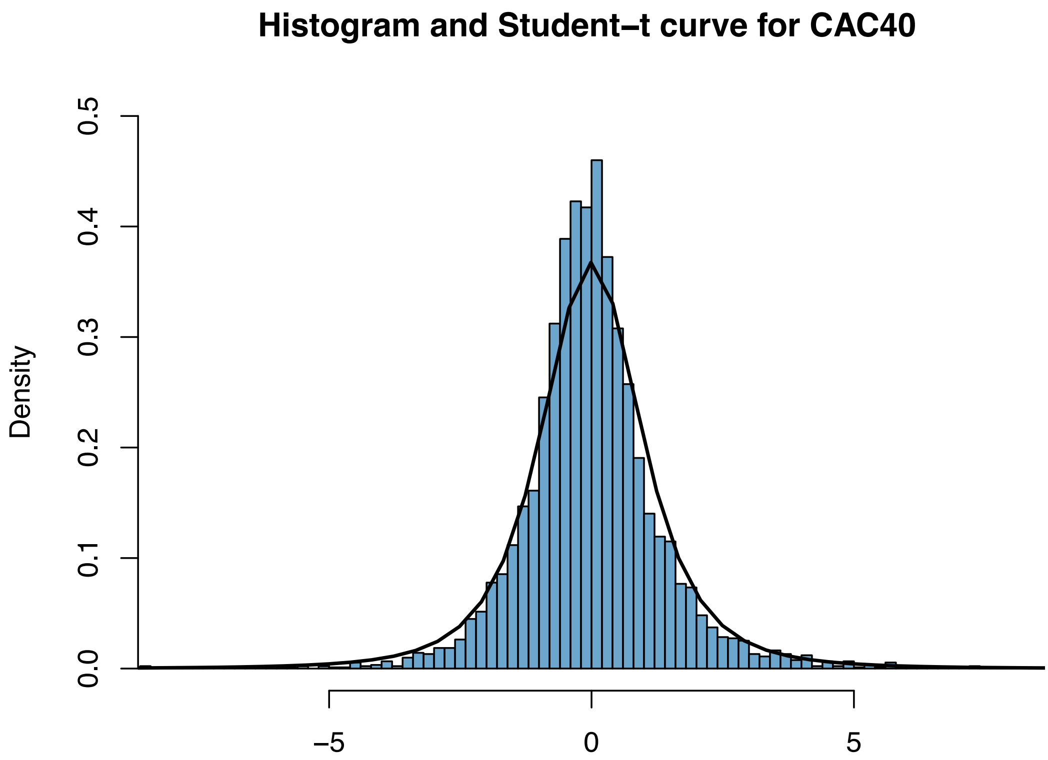

Figure 23 Histogram and Student's t curve for CAC 40

FTSE MIB analysis will follow the same steps of CAC 40. We will analyze Q-Q plot of FTSE MIB against student-t with 3.2 DoF and then we focus on histogram visualization.

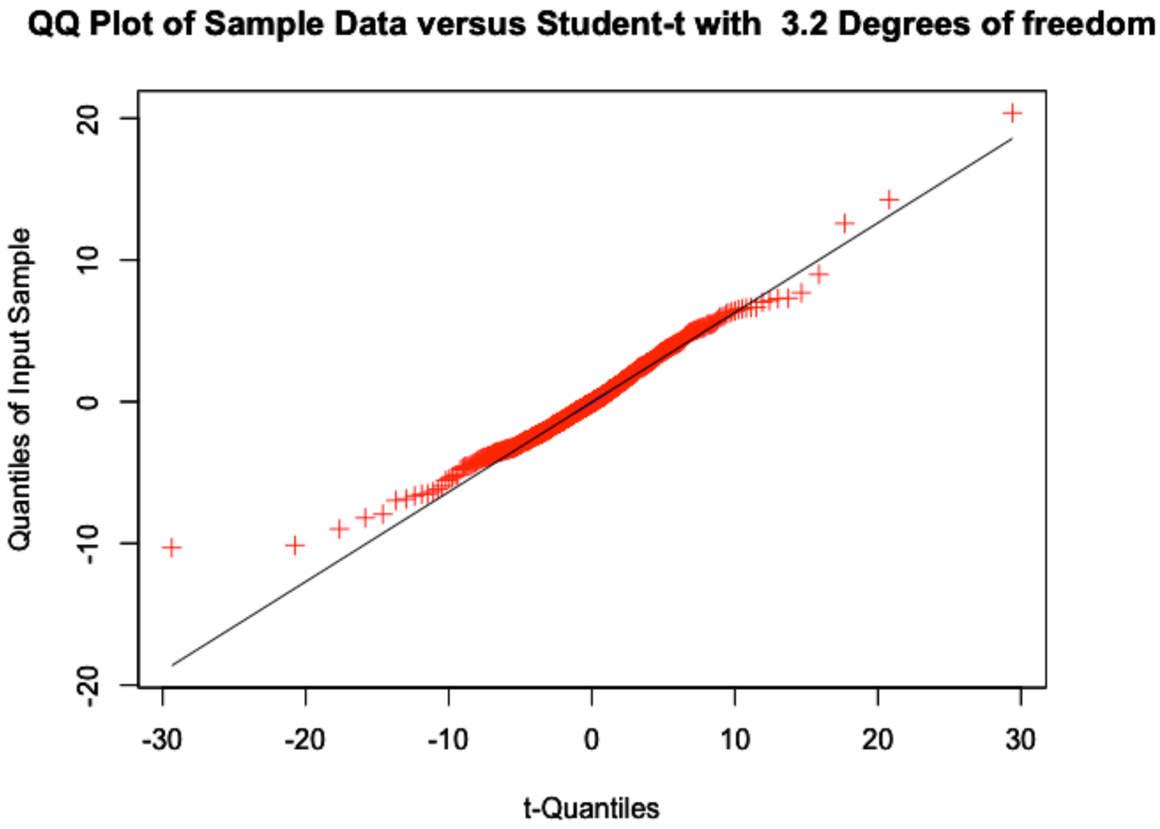

Figure 24 Student's t QQ-plot for FTSE MIB

As observed before, student's t Q-Q plot represents a better fit for our data. This is true also for FTSE MIB, we can notice how returns align to student's t line indicating a better fit compared to normal line.

The histogram below is another graphical confirmation of student's t good fitting for FTSE MIB returns. In fact, histogram bars rarely overcome student's t line confirming our presage about the distribution nature.

These visual representations confirm that student's t with 3.2 DoF optimally describes FTSE MIB returns.

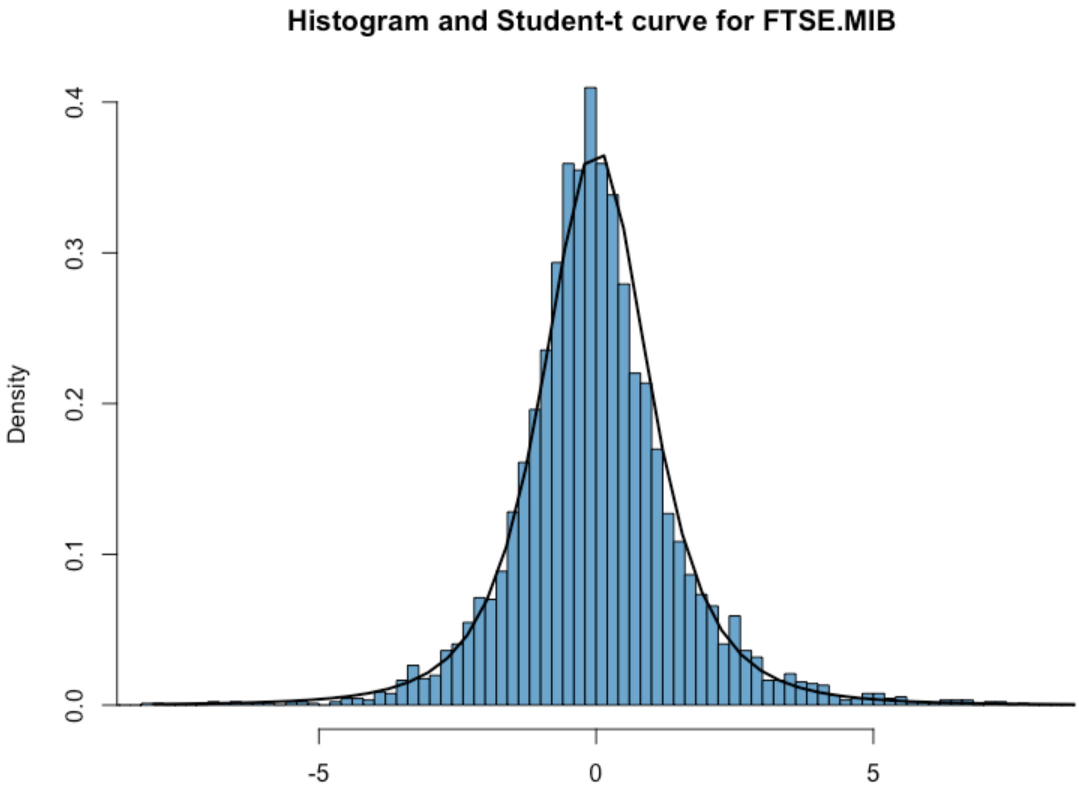

Figure 25 Histogram and Student's t curve for FTSE MIB

Extreme value Theory: CAC 40 and FTSE MIB analysis

In the last paragraph we established that our data cannot be described by normal distribution. In fact, Student's t looks as improvement for both CAC 40 and FTSE MIB. Now, we want to investigate how the two distributions behave in their tails, particularly concerning about extreme losses. In financial market tail event can be disruptive and threatening market stability. As showed in the previous chapter Extreme Value Theory (EVT) is a crucial analytical tool that allows to study returns distribution tails. In our CAC 40 and FTSE MIB analysis we want to estimate and model the impact of certain rare events on the market and on the two indexes. Our approach involves determining optimal fit for our data using Block Maxima method and Peaks over threshold. Our analysis ends with a focus on risk measurement, to estimate likelihood and magnitude of extreme losses.

As we are studying loss distribution we look at negative returns. R language allows us to find optimal fit for both Extreme value theory applications. Firstly, we are going to show Block Maxima method estimations and subsequently explaining Peaks over the threshold findings.

Block Maxima analysis

As explained in the previous chapter, we are going to divide our data into blocks. Each block has one month size, thus having a maxima observation for each month from 2004 to 2022. In the plot below we can notice how maxima from each distribution are distributed in similar positions, highlighting correlation between the two indexes.

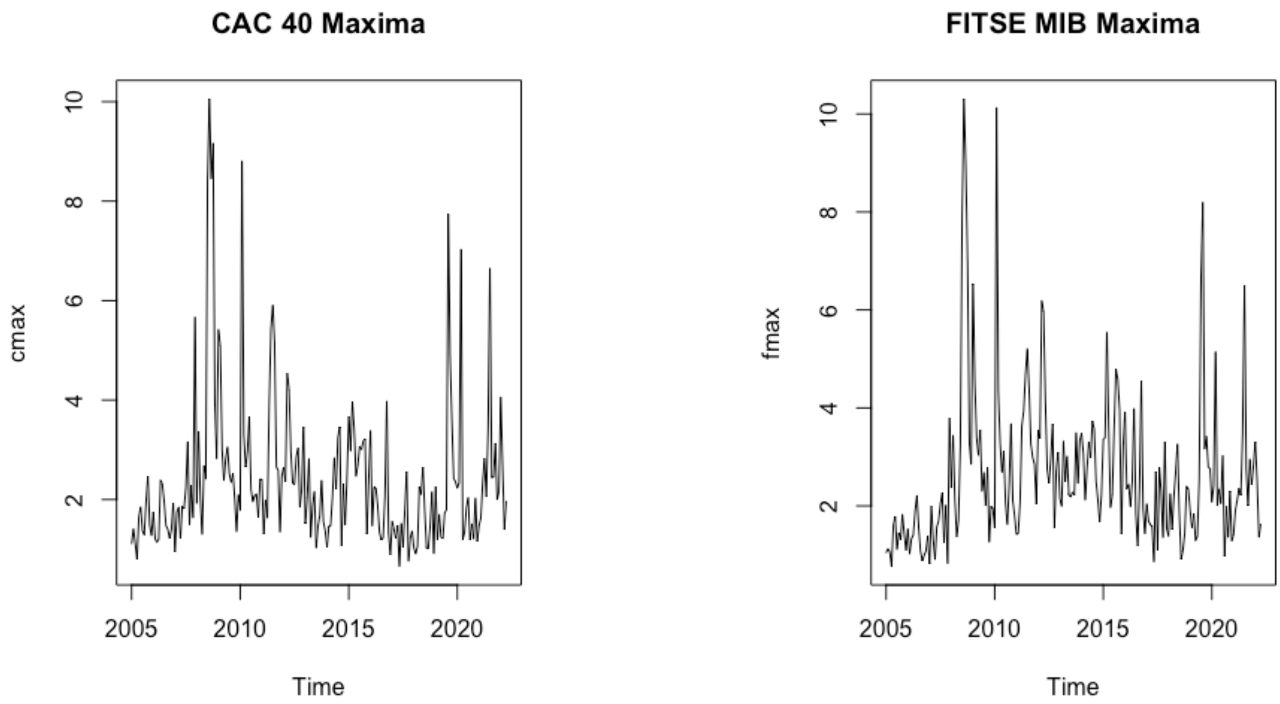

Figure 26 CAC 40 and FTSE MIB monthly maxima

In the table below we can analyze BM parameter estimates for CAC 40 loss distribution. Parameter estimates are obtained using *gevdist* (ML estimation in R). Shape parameter ξ is equal to 0.2973, determining a Fréchet distribution. This kind of distribution is characterized by fat tails, implying higher probabilities for extreme losses. Location parameter u is around 1.7, while scale parameter β is around 0.73.

BM: CAC 40	COEFFICIENTS	STANDARD ERROR
$\boldsymbol{\xi}$	0.2973228	0.004187933
$\boldsymbol{u}$	1.6987367	0.003997467
$\boldsymbol{\beta}$	0.7299249	0.003361472

Table 9 BM coefficients for CAC 40

Analyzing residuals is crucial to understand if results above are optimal estimates for our losses distributions. Goodness of fit is basilar to correctly studying loss distribution and computing risk measure. GEV residual histogram has an exponential shape suggesting we have obtained optimal parameters. Additionally, the scatterplot of residuals reinforces our idea since there is no clear pattern in residuals. Once again Q-Q plot of residuals confirms our intuition, as our data perfectly align to exponential quantiles. All the graphs indicate perfect fitting, meaning that our estimates seem reliable.

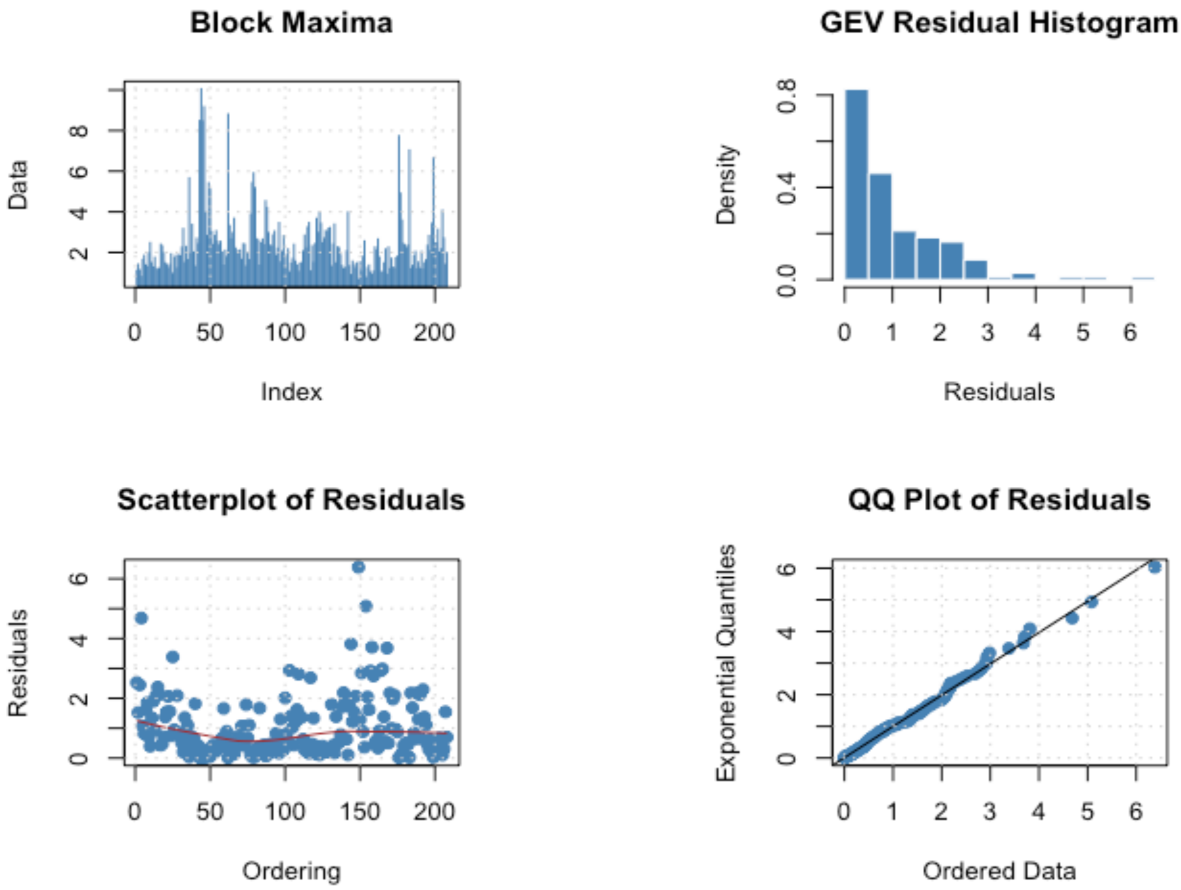

Figure 27 CAC 40 residuals

Now, we analyze BM parameter estimates for FITSE MIB. Similar to CAC 40, shape parameter (ξ) suggest we are facing a Fréchet distribution. Location parameter u is around 1.9, while scale parameter β is around 0.88.

BM: FTSE MIB	COEFFICIENT	STANDARD ERROR
$\boldsymbol{\xi}$	0.216673	0.004225543
$\boldsymbol{u}$	1.917174	0.004837545
$\boldsymbol{\beta}$	0.875330	0.003884456

Table 10 BM coefficients for FTSE MIB

Even for FITSE MIB our fit seems optimal. By checking at residuals below we can state goodness of fit for BM parameter estimates. In fact, they follow similar CAC 40 pattern.

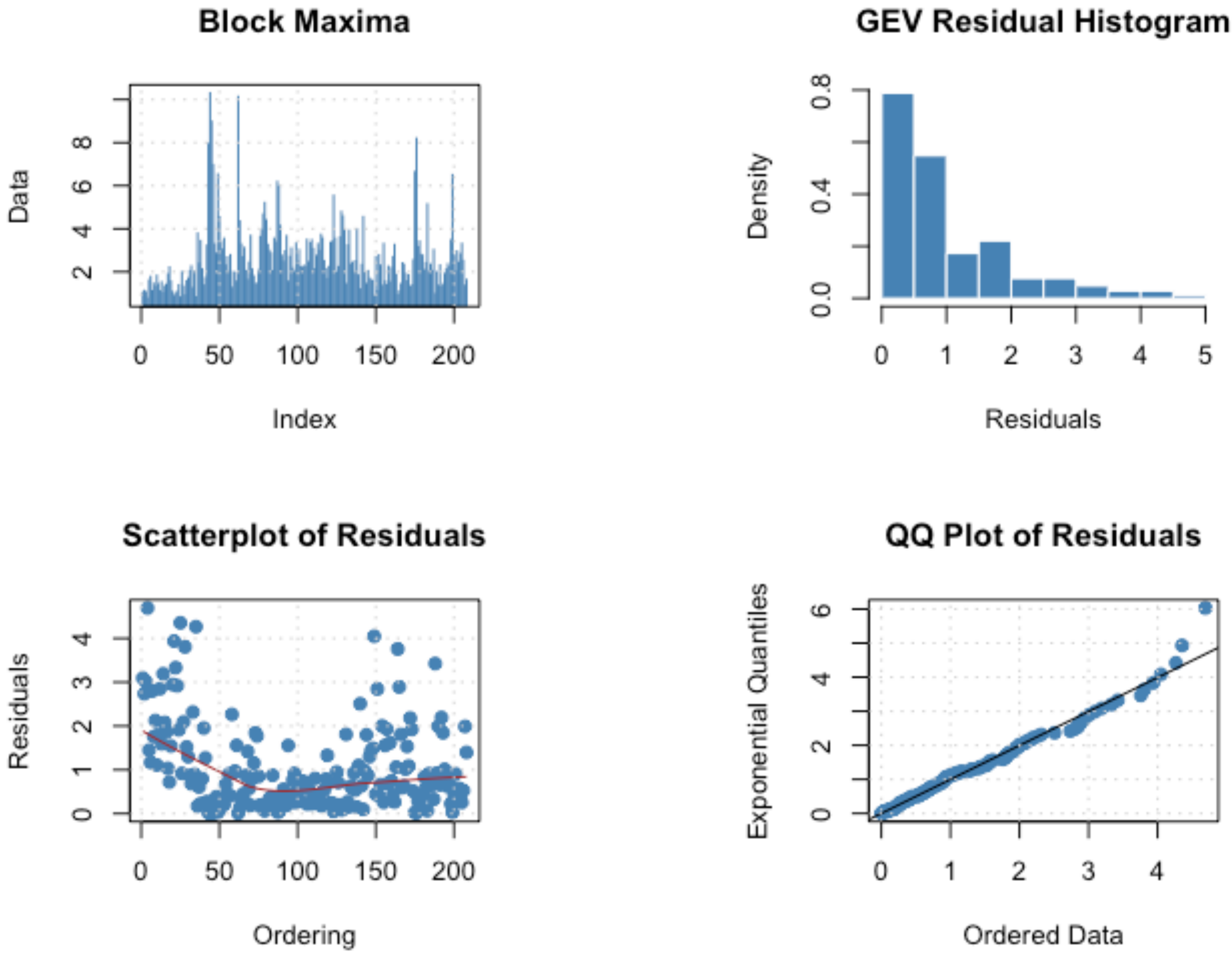

Figure 28 FTSE MIB Residuals

Now we are going to compute some risk measures. While we know that maximum loss observed is 10.05% and 10.30% respectively for CAC 40 and FITSE MIB, our analysis extends beyond these values. Our objective is to recognize the probability of experiencing similar losses in the future and the frequency with which such events occur.

A crucial risk measure is probability to observe a loss in the next block larger than max loss observed. For example, in the CAC 40 case, we want to know what is the probability to observe a loss L higher than 10,05% during the next month. To do that we compute P(L > 10,05%), which is equal to 1 minus the probability to observe a loss equal to 10,05%.

Another fundamental risk measure is return level. The kn-block return level can be interpreted as "*the level that is exceeded once out of every* kn*-blocks on average. Formally, the return level* $R_{n,k}$ *is such that:*

$$Pr\left(M_n > R_{n,k}\right) = \frac{1}{k}$$

Note that by construction $R_{n,k}$ *is the* $(1 - 1/k)$ *quantile of* $H_\xi(\mu, \sigma)$*, and then:*

$$R_{n,k} \approx H_{\xi,\mu,\sigma}^{-1}\left(1 - \frac{1}{k}\right) = \mu - \frac{\sigma}{\xi}\left(1 - \left(-ln\left(1 - \frac{1}{k}\right)\right)^{-\xi}\right)$$

The n*-block in which the return level is exceeded is called a stress period*".[24] It represents the maximum expected loss in a given time period. For instance, computing which the max loss expected in the next year will be.

The kn-block return period, $k_{n,u}$ is the last risk measure we are going to see regarding Block Maxima method. It is defined as the "*average number of blocks we must wait before we observe the extreme event* $M_n > u$*, for a given* u*.*

Since $H_{\xi,\mu,\sigma} = Pr\ (M_n < u)$*, then the event* $Y = (M_n > u)$ *is observed the first time after* κ *blocks has probability* $Pr\ (Y = \kappa) = H_{\xi,\mu,\sigma}^{\kappa-1}\left(1 - H_{\xi,\mu,\sigma}\right)$*. This is a geometric distribution with average* $1/1 - H_{\xi,\mu,\sigma}$*. Therefore the* k*-block return period is*[25]:

$$k_{n,u} = \frac{1}{1 - H_{\xi,\mu,\sigma}(u)}$$

For example, we can compute how many months we must wait before observing a loss higher than the previous maximum loss.

In the table below we have summarized results for each of the risk measures previously explained.

[24] Formulae taken from Professor Raggi course on Risk Management.

[25] Definition and formula taken from Professor Raggi course on Risk Management.

TEST	CAC 40	FITSE MIB
MAXIMUM LOSS	10.05267%	10.30386%
PROBABILITY OF EXCEEDANCE	0.6814%	0.558%
K-BLOCK RETURN PERIOD (IN MONTHS)	146.75	179.22
K-BLOCK RETURN LEVEL (1 YEAR)	4.3176	4.7343

Table 11 Risk measures (BM)

Comparing CAC 40 and FITSE MIB results reveals insightful patterns but also distinctions. Despite both stock indexing share a maximum observed loss of around 10%, the probability of exceedance slightly differs, standing at approximately 0.68% for CAC 40 and 0.56% for FITSE MIB. This difference in probabilities reflects various market dynamics of each stock index.

The k-block return periods computed on maximum loss observed are around 147 months for CAC 40 and 179 months for FITSE MIB suggesting a large difference in frequencies of extreme events. For CAC 40 we expect a loss equal or higher than 10.05% in 12 years, while for FITSE MIB we expect a loss equal or higher than 10.30% in 15 years. A loss around 10% is an extreme event that happens once in more than 10 years, thus we are also interested in analyzing more frequent events. To do that we can compute k-block return level within 1 year. In this case the k-block return levels are 4.32% for CAC 40 and 4.73% for FITSE MIB over a 12-month period, signify distinct expectations for the maximum losses within a year. In both case we observe a probability equal to 8,33%. Another method to study probability of more frequent events is to define a determined loss and then computes risk measures seen above. There is a probability of 5,53% to observe a loss of 5% for CAC 40 and 7,04% for FITSE MIB. K-block return period for CAC 40 is 18 months, whereas FITSE MIB return period is 14 months.

Risk measures show how FITSE MIB can be considered a riskier asset if compared to CAC 40.

Peaks over threshold analysis

After having analyzed how Block maxima performs on data and, on that basis, risk measures are computed, we are going to show Peaks over thresholds (POT) results. To determine an optimal threshold for both stock indexes we display mean-excess plots and find where $\hat{e}(u)$ begins to be linear. In the section below we show mean-excess plot for each stock index and decide the optimal threshold u.

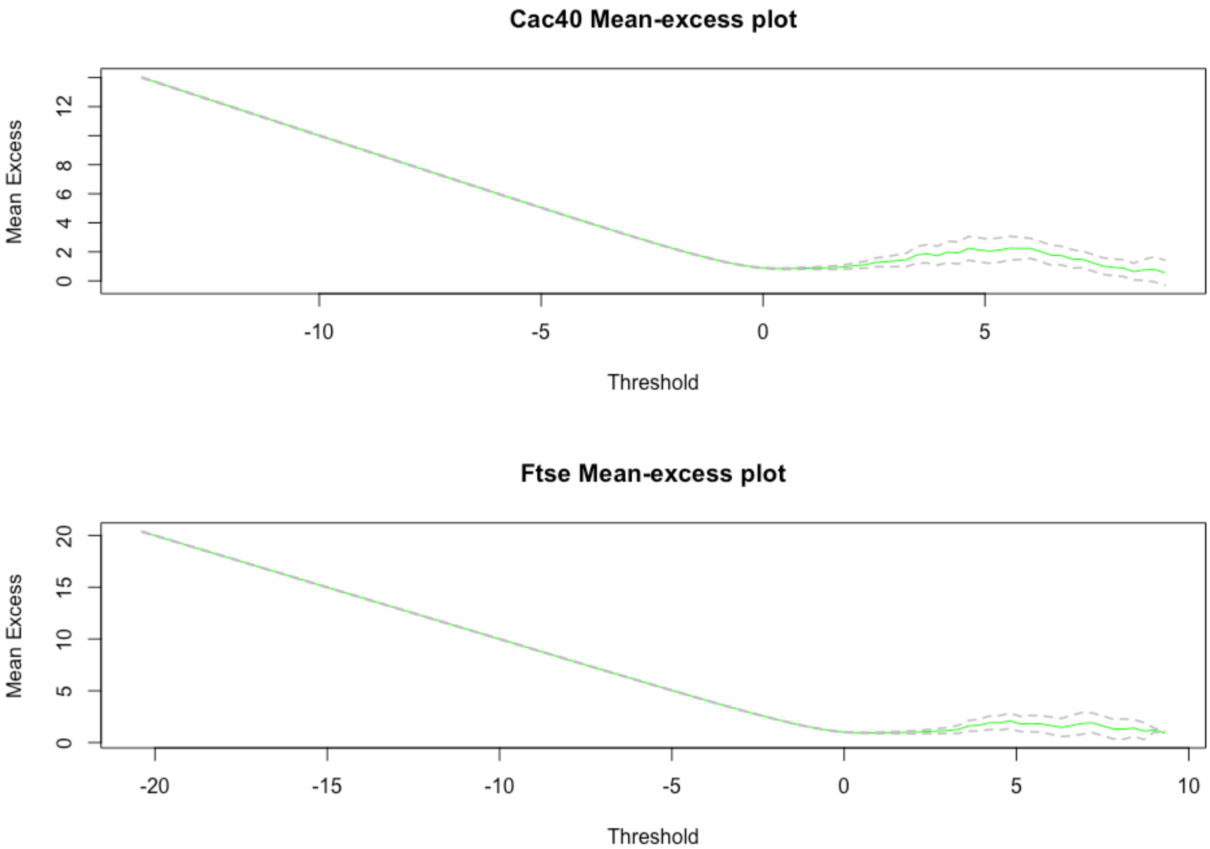

Figure 29 Mean Excess plot for CAC 40 and FTSE MIB

Analyzing CAC 40 Mean-excess plot we can conclude that optimal threshold could be around 1.5 or 2, whereas observing FTSE MIB optimal threshold u could be around 2 or 2.5.

After deciding optimal u, we can find POT best fit for our losses. By computing Maximum likelihood estimates, we obtain the following parameter estimates:

POT: CAC 40	COEFFICIENT	STANDARD ERROR
$\hat{\xi}$	0.1901761	0.0008146189
u	1.5	
$\hat{\beta}$	0.7491697	0.0008173213

Table 12 POT coefficients for CAC 40

POT: FTSE MIB	COEFFICIENT	STANDARD ERROR
$\hat{\xi}$	0.2036028	0.00100696
u	2	
$\hat{\beta}$	0.7742930	0.001010949

Table 13 POT coefficients for FTSE MIB

CAC 40 shape parameter $\hat{\xi}$ estimate indicates that CAC 40 losses are heavy tailed. We can notice as also FTSE MIB shape parameter $\hat{\xi}$ suggests higher kurtosis for the Italian stock index distribution as well.

As usual, our analysis is accompanied by visual representations that help to visualize our intuitions. In this paragraph we observe histograms of our returns with fitted GPD line. First histogram represents CAC 40 loss distribution, while FTSE MIB loss distribution is represented just below.

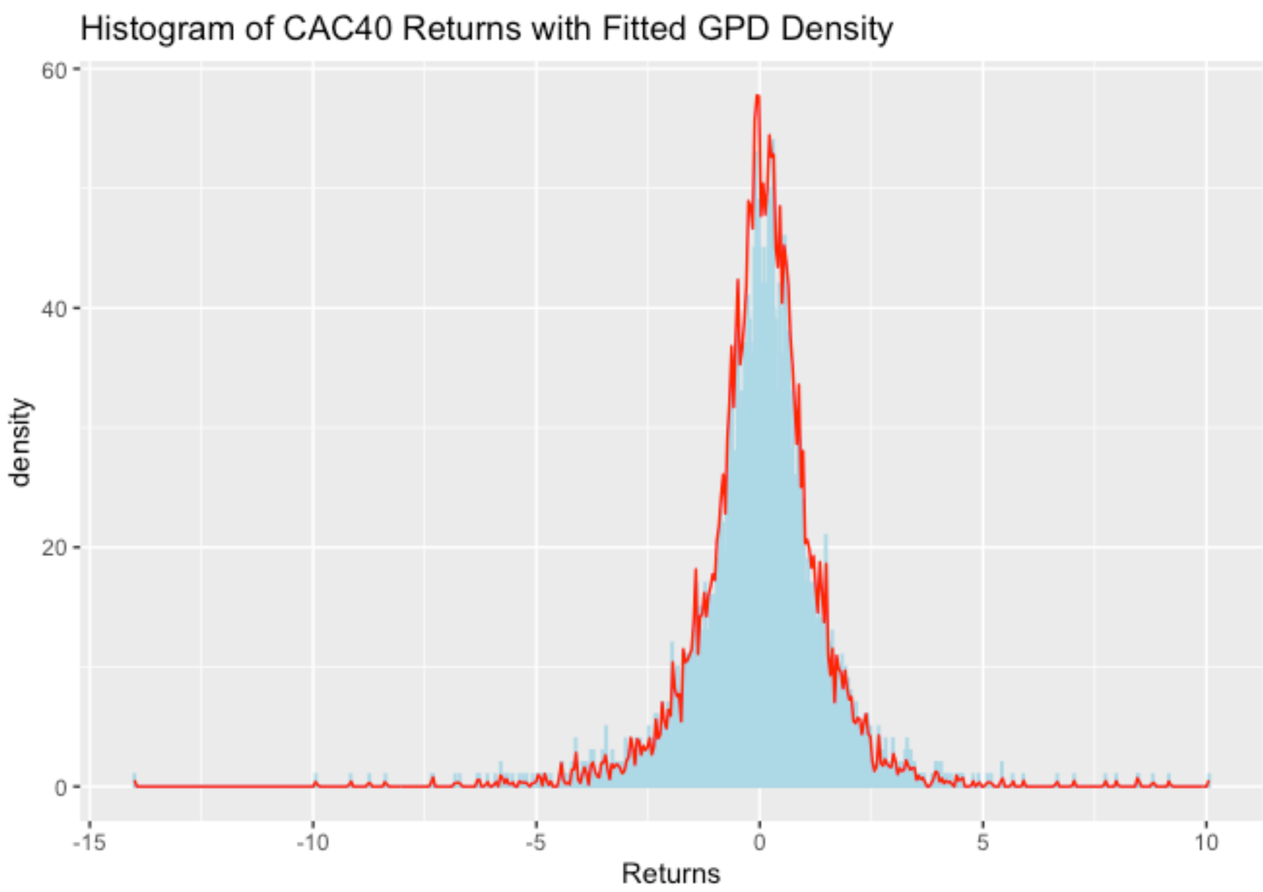

Figure 30 Histogram with fitted GPD density of CAC 40

By observing CAC 40 histogram we can notice that GPD properly fits the data. CAC 40 losses are well approximated below GPD line. We observe similar results in FTSE MIB histogram, even if in some area losses are not perfectly distributed. For instance, histogram central part does not fit optimally GPD line.

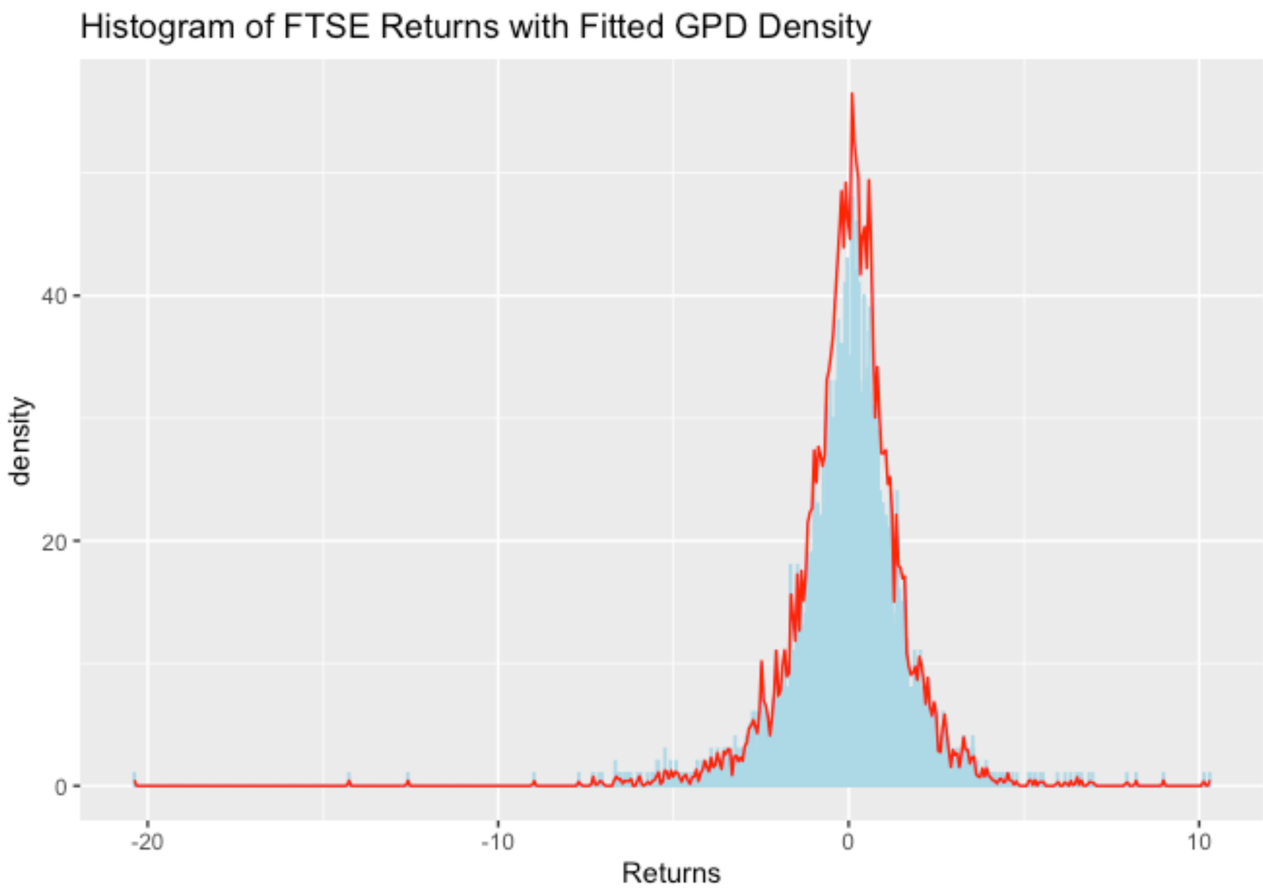

Figure 31 Histogram with fitted GPD density of FTSE MIB

At first glance we can state that POT method appears to be an optimal solution to model FTSE MIB and CAC 40. To confirm this idea, we will conduct a detailed analysis on residuals mirroring the approach used for BM method. Residuals analysis aims to understand POT suitability and reliability for these stock indexes.

Analyzing residuals is crucial to understand if results above are credible enough for the losses distributions. Goodness of fit is key to correctly study loss distribution and compute risk measure. GEV residual histogram has an exponential shape suggesting we have obtained optimal parameters. Additionally, the scatterplot of residuals reinforces our idea since there is no clear pattern in residuals. Once again Q-Q plot of residuals confirms our intuition, as our data perfectly align to exponential quantiles. All the graphs indicate perfect fitting, meaning that our estimates seem reliable

In the next paragraph we observe excess distribution and tail of underlying distribution, while just below there are residuals plot. We start by analyzing CAC 40 graphs.

In the top left plot are depicted losses over threshold against GPD distribution. Excess distribution follows GPD distribution indicating optimal fit. On the top right graph, we observe how CAC 40 tail fit GPD distribution. Given that only few observations don't follow GPD line we can state that CAC 40 tail are well described by GPD model. Moving to residuals we can notice that in scatterplot of the residuals lack of any clear pattern. residuals quantiles align quite well against exponential quantiles in the Q-Q plot. Considering these signals, we can state that GPD represents a valid solution to model CAC 40 losses.

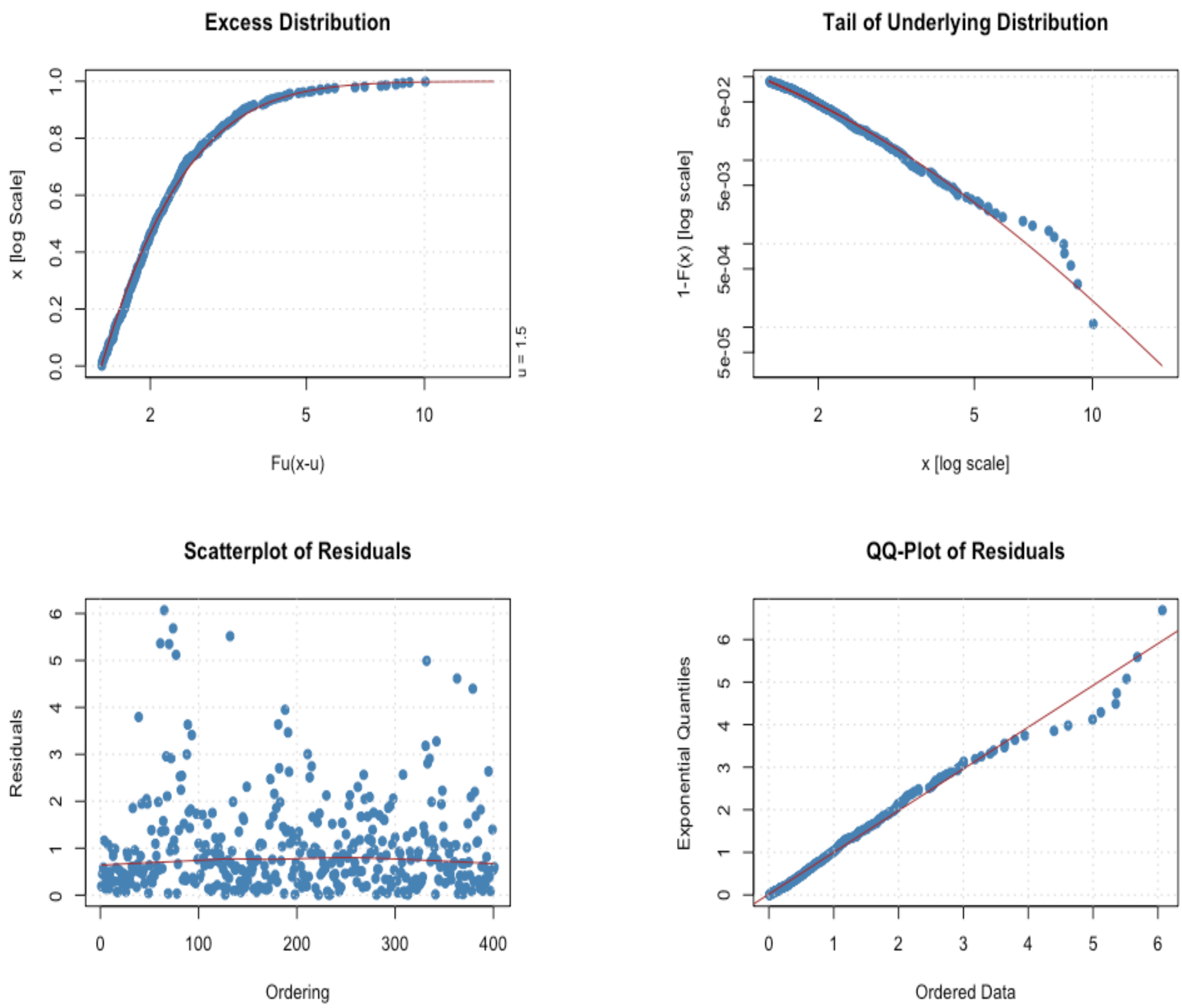

Figure 32 GPD Residuals for CAC 40

FTSE MIB plots indicate that GPD model provides an accurate representation of FTSE MIB losses. In fact, excess distribution follows GPD distribution, while we observe GPD goodness of fit in the top right graph. Even residuals confirm our hypothesis, no patterns in the scatter-plot and residuals quantiles perfectly align with exponential quantiles.

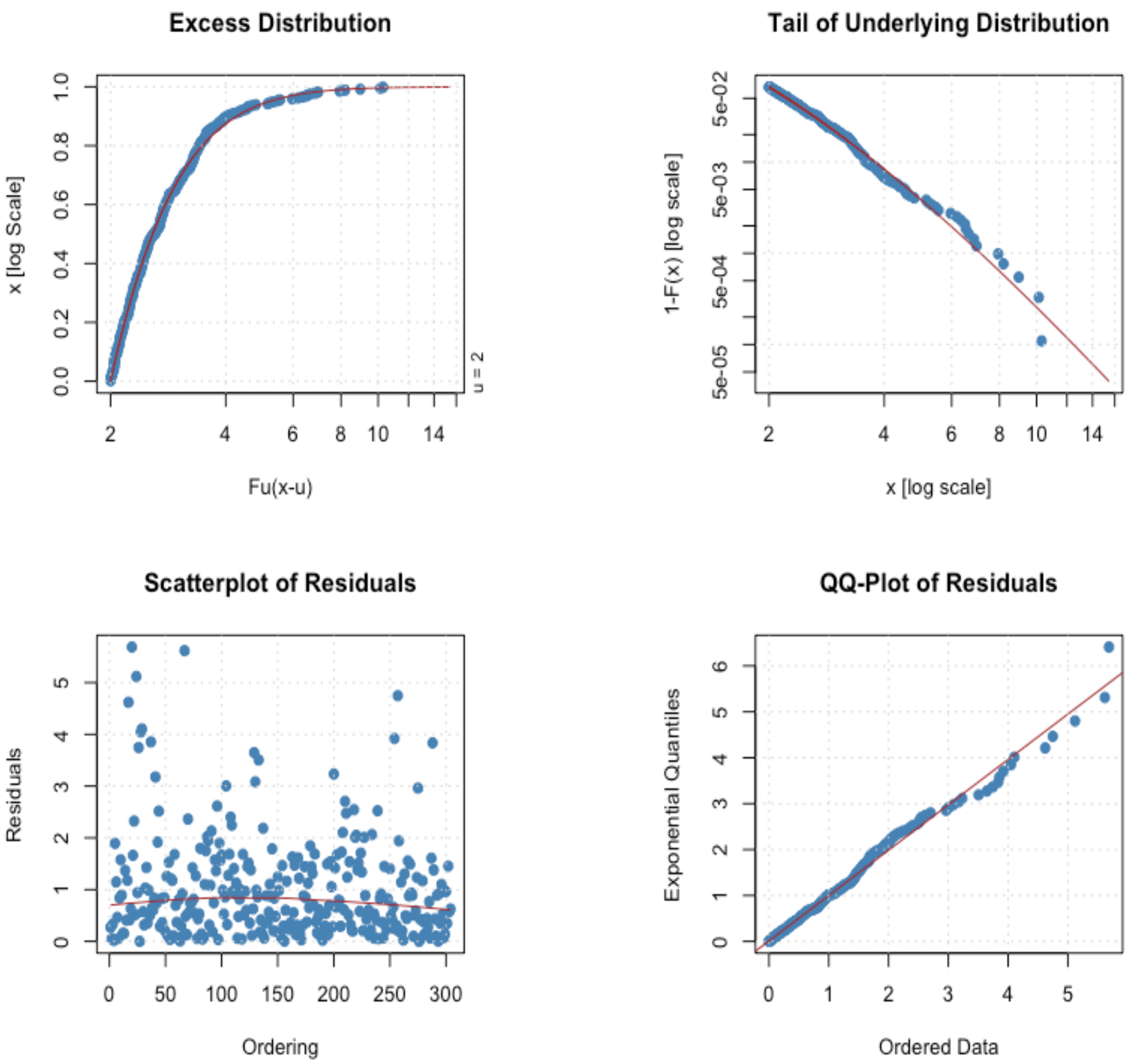

Figure 33 GPD Residuals for CAC 40

Risk measures: VaR_α and ES_α

We found optimal fit both for CAC 40 and FTSE MIB in POT context. Now we are interested in analyzing potential extreme losses. To do that we use VaR_α and ES_α choosing different α. A higher α corresponds to more extreme losses with decreased probability. We compute VaR_α and ES_α following the same procedure used in Google example.

CAC 40 (α)	$\mathbf{VaR}_\alpha$	$\boldsymbol{ES_\alpha}$
0.95	1.94%	2.97%
0.99	3.51%	4.91%
0.999	6.78%	8.95%

Table 14 VaR_α and ES_α for CAC 40

The table above show CAC 40 VaR_α and ES_α estimates. $\mathrm{VaR}_{0.95}$ indicates a loss equal or higher than 1.94% with 5% probability. While average loss if $\mathrm{VaR}_{0.95}$ is exceeded is $ES_{0.95}$ =2.97%. As explained before as the coverage level increases, we observe higher losses with lower probability.

On the CAC 40 histogram below are depicted $\mathrm{VaR}_{0.95}$ and $ES_{0.95}$.

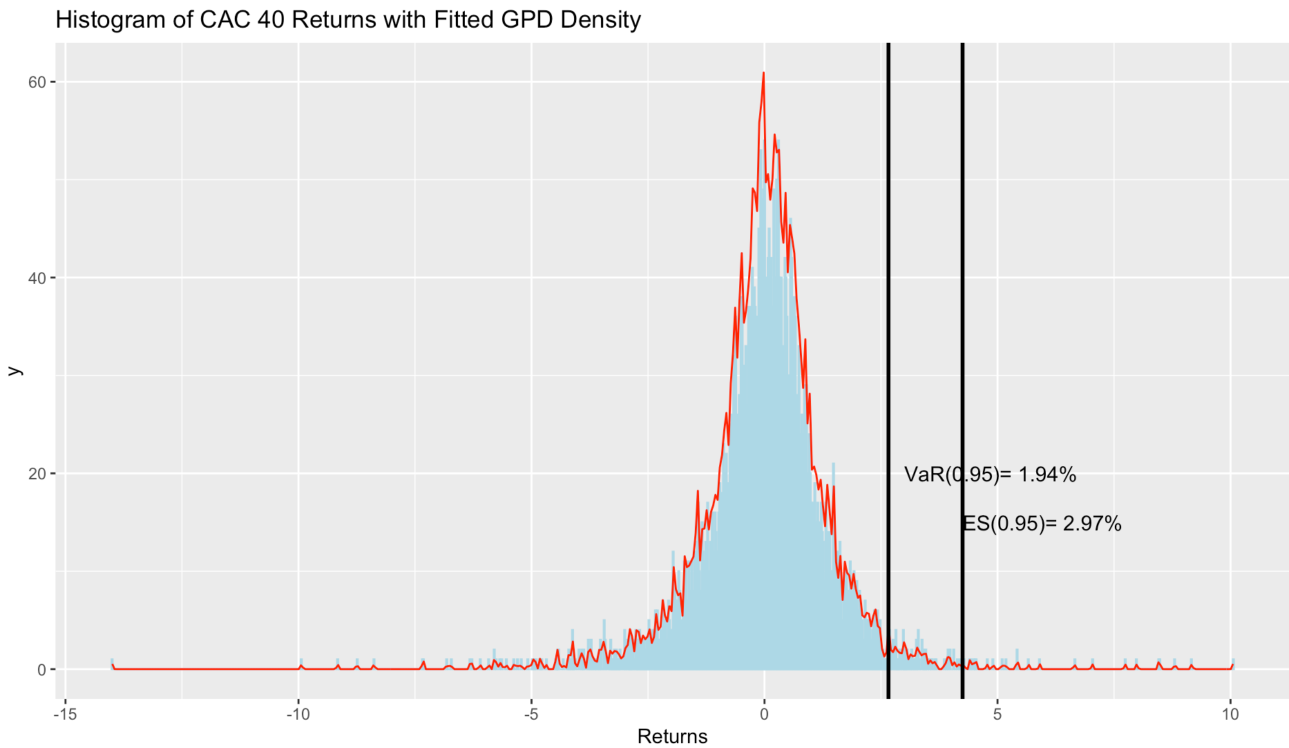

Figure 34 Histogram of CAC 40 with plotted $VaR_{0.95}$ and $ES_{0.95}$

In the table below VaR_α and ES_α estimates for FTSE MIB are plotted. We observe slightly higher losses if compared to CAC 40 estimates.

FTSE MIB (α)	$\mathbf{VaR}_\alpha$	ES_α
0.95	2.24%	3.27%
0.99	3.78%	5.22%
0.999	7.13%	9.41%

Table 15 Table 13 VaR_α and ES_α for CAC 40

As already showed in the CAC 40 example, the plot below depicts FTSE MIB histogram with $\mathrm{VaR}_{0.95}$ and $ES_{0.95}$.

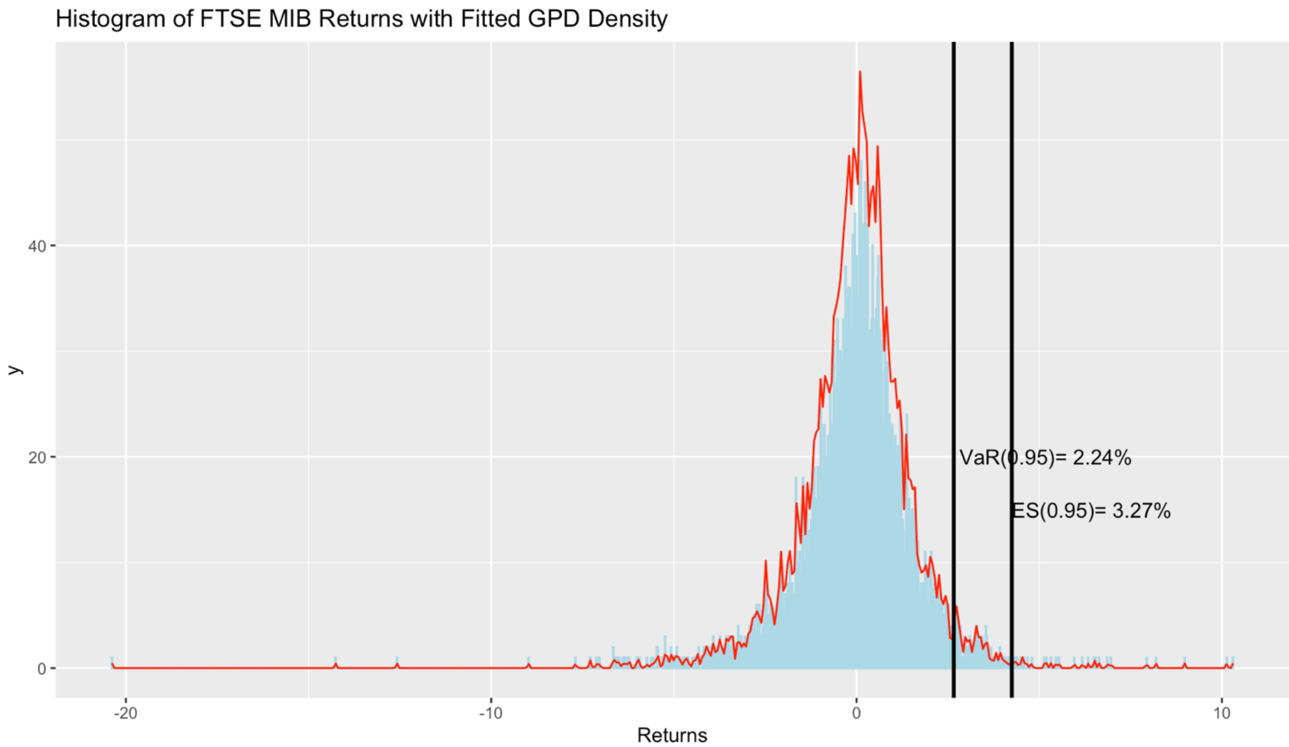

Figure 35 Figure 35 Histogram of FTSE MIB with plotted $VaR_{0.95}$ and $ES_{0.95}$

Copula: CAC 40 and FTSE MIB analysis

In this paragraph our aim is to create copulas that optimally capture interdependences between CAC 40 and FTSE MIB. We will use two different approaches. Firstly, we build copulas using marginals obtained in the previous analysis, such as Student's t, POT and BM estimates. The copula is estimated by using maximum likelihood estimation to obtain the parameters. In the second case we directly fit the optimal copula on our return time series using the library *Copula* in R.

After introducing possible copula families, we show our fit results. Based on them, we choose some copulas which will help us analyze correlation between the two indexes.

Our marginal models estimates are summarized below:

STUDENT T	CAC 40	FTSE MIB
MEAN	-0.05515526	-0.05302316
STANDARD DEVIATION	1.47670674	1.64009850
DEGREE OF FREEDOM	3.00482283	3.20365077

BM ESTIMATES	CAC 40	FTSE MIB
ξ	0.2973228	0.216673
u	1.6987367	1.917174
β	0.7299249	0.875330

POT ESTIMATES	CAC 40	FTSE MIB
$\hat{\xi}$	0.1901761	0.2036028
u	1.5	2
$\hat{\beta}$	0.7491697	0.7742930

Table 16 parameter estimates summary

From these marginals, we create three different copulas by using R language in our estimation process.
Rotated Tawn type 1 90 degrees is derived from Student's t marginals, while BM and POT marginals respectively generate a Tawn type 2 and a Frank copula. Lastly, we create a copula by fitting the optimal copula on our return time series. Parameter estimates for each copula are summarized below:

COPULA FAMILY	1° PARAMETER	2° PARAMETER	KENDALL'S TAU
BB1	0.81	2.24	0.68
TAWN (TYPE 2)	3.21	0.78	0.57
FRANK	-0.38	-	-0.04
TAWN (TYPE 1) 90	-13.36	0.00	0.00

We have already introduced BB1 and Frank copula in previous chapter, while Tawn copula refers to a particular kind of copula that describes dependence structure characterized by asymmetric dependency. Tawn copula is a particular version of the Gumbel copula, in which two dependency parameters are added. This copula is an extension of the Gumbel copula because the new parameters allow to describe either upper or lower tails dependencies. In R, Tawn copula is defined in two versions: each one has one parameter equals to 1. Thus, Tawn type 1 refers to right skewed Tawn, while Tawn type 2 is characterized by heavier left tail.[26]

We are interested to know which copula is an optimal representation of CAC 40 and FTSE MIB interdependence. The goodness of fit results are depicted below:

COPULA FAMILY	LOGLIKELIHOOD	AIC	BIC
BB1	3519.04	-7034.08	-7021.22
TAWN (TYPE 2)	120.13	-236.27	-229.59
FRANK	2	-1.99	2.91
TAWN (TYPE 1) 90	-13.36	0.00	0.00

Table 17 Copula family fit estimates

Analyzing this table allow us to choose between various copula families which represents the best fit for our data. BB1 copula stands out as the best fit, it has the higher loglikelihood while having lower values for both AIC and BIC, ensuring the best tradeoff between fit and model complexity. On the other hand, both Frank and Tawn type 1 copula show poor fit with value around 0. We also consider Tawn type 2 as a good possible model to shape correlation between the two stock indexes as it shows higher loglikelihood and lower AIC/BIC value if compared with the other two copulas.

Considering goodness of fit we can relate on the first two copula models depicted in the table below. BB1 is a flexible kind of copula allowing to model both left and right tail dependence, in this case our parameter estimates suggest that we can observe more events in left tail.

[26] https://cran.r-project.org/web/packages/VineCopula/readme/README.html

Tawn type 2, which is the second model chosen for our analysis, suggests similar behaviour for the two stock indexes. In the next paragraph we are going to expand this analysis also considering copulas graphical representation.

We chose to start by analyzing the BB1 copula since it represents optimal one for thhe stock indexes. BB1 Copula allows to model various kind of interdependence thanks to its flexible approach. As already stated in chapter 1, BB1 copula can be used to described copula affected by either right or left tail dependence.

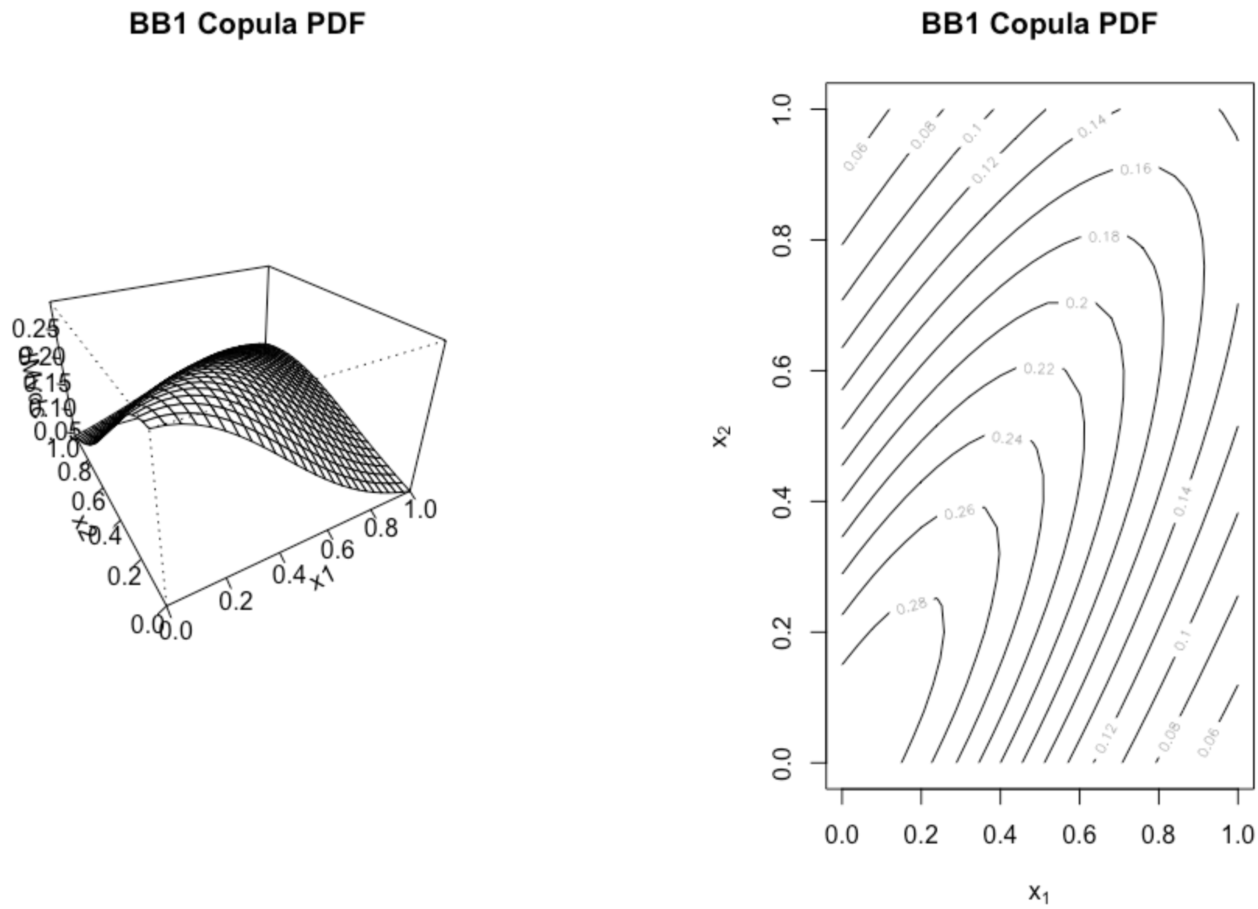

Figure 36 BB1 Copula PDF (ϑ=0.81, δ=2.24, tau=0.68)

The BB1 copula is created by directly applying optimal fitting copula to the data. Its PDF suggests that correlation structure between CAC 40 and FTSE MIB can be modeled with an asymmetric tail heaviness. In fact, this copula shows slightly left tail dependence, denoting asymmetry regarding dependence structure. Left tail dependence means that there is higher likelihood of extreme positive co-movement compared to co-movement in the right tail. Thus, we expect to observe more positive returns than losses. We can notice a central cluster signaling that the two stock indexes move together. However, this cluster decrease while we analyze more extreme losses.

Tawn copula is built merging two BM marginals distribution. Both marginals distributions are Fréchet distribution, thus showing heavy tails.

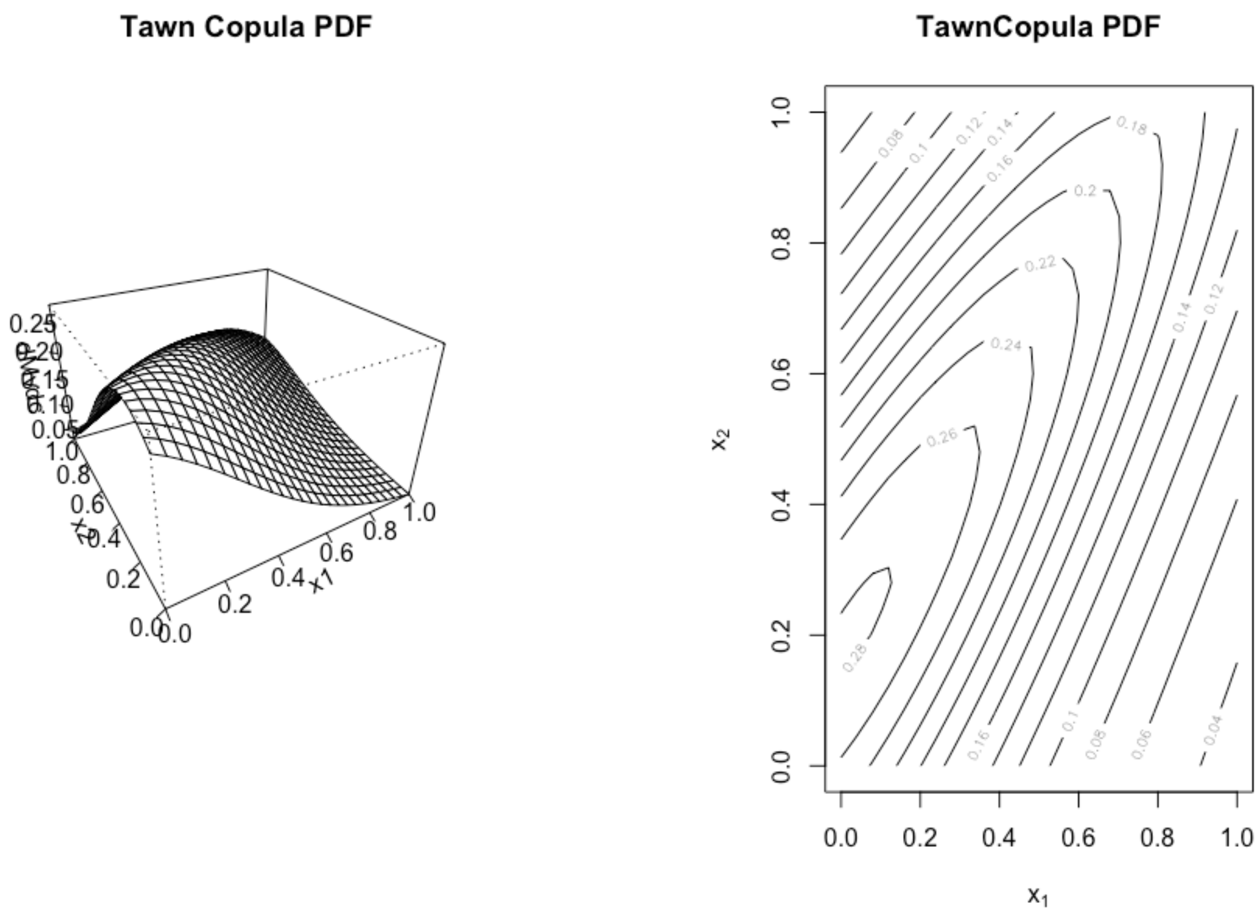

Figure 37 Tawn type 2 Copula PDF (1st parameter= 3.21; 2nd parameter= 0.78 tau=0.57)

Tawn type 2 Copula PDF highlights asymmetric dependence structure between the two stock indexes. In fact, we can notice a higher concentration in the left tail compared to the right one. Left tail dependence, observed both in BB1 copula and Tawn type 2 copula, seems to be a characteristic of CAC 40 and FTSE MIB correlation structure.

If we compare the two copulas, we can spot a difference in the dependence structure: Tawn Copula shows distribution cluster on the center-left rather than on the center. The main difference between these two copulas is that probably for Tawn Copula model FTSE MIB experiences more extreme losses than CAC 40, thus exhibiting left asymmetry.

4. The role of Copula in stress testing: analysis on CAC 40 and FTSE MIB

To conclude this analysis, we introduce a new concept in this thesis: the stress testing. Widely applied by regulators and financial actors, stress testing is a fundamental concept in risk management. In fact, stress tests help regulators and financial firms to assess potential losses in different scenarios. On this purpose, we can cite Bank for International Settlement (BIS):

"Stress tests are forward-looking exercises that aim to evaluate the impact of severe but plausible adverse scenarios on the resilience of financial firms. They involve the use of models and data at the firm or system-wide level and may rely on historical or hypothetical scenarios."[27]

They give the opportunity to simulate eventual losses under various scenarios and verify if a financial firm capital is enough to cover for potential losses. Flexibility and data driven analysis represent its perks, while its main limitation depends on the fact that we do not know with which probability the loss will happen.[28] As a way to address this weakness we decide to implement a specific type of stress loss, by considering copula in the analysis. Our stress test is based on our two stock indexes CAC 40 and FITSE MIB. In the table below we summarize various information both regarding portfolio characteristics and risk related to portfolio itself. Regarding risk measure we compute stress move and stress loss and related probability to happen. Obviously, to have an optimal view on the risk, we integrate VaR_α and ES_α in our stress test.

We assume an investment of 100 000€ for each stock indexes with December 2023 average price chosen as stock indexes price. To compute stress move we set a loss equal to 5% and a loss equal to 10%, we investigate which is the probability to simultaneously observing these kinds of losses. We use Empirical cumulative distribution function (ECDF) on the two time series and CDF on BB1 copula to compute likelihood of loss happening, while we used POT estimates to compute VaR_α and ES_α.

STOCK INDEX	PRICE	PORTFOLIO VALUE	STRESS MOVE	STRESS LOSS	PROBABILITY
CAC 40	7420	100 000	5%	5000	0.23%

[27] https://www.bis.org/fsi/fsisummaries/stress_testing.pdf

[28] https://www.bis.org/fsi/fsisummaries/stress_testing.pdf

FTSE.MIB	30440	100 000	5%	5000	

Table 18 Stress test (stress loss equal to 5%)

In our stress testing scenario, loss is equal to 5000€ for each stock index determining a 10000€ total portfolio loss. We are interested in discovering with which probability we will observe a simultaneous plunge by 5% in CAC 40 and FTSE MIB price. A 5% decrease in both stock indexes has probability equal to 0.23%. As stated before, we are going to use ECDF and copula CDF in this process. To compute probability that stress losses happen simultaneously in both stock indexes we adapt the following formula:

$$\begin{aligned} P(M_1 > x_1, M_2 > x_2) \quad &= 1 - P(M_1 \leq x_1) - P(M_2 \leq x_2) + P(M_1 \leq x_1, M_2 \leq x_2) \\ &= 1 - H_1(x_1) - H_2(x_2) + H_\infty(x_1, x_2) \\ &= 1 - H_{\text{CAC 40}}(x_1) - H_{\text{FTSE.MIB}}(x_2) + C\big(H_{\text{CAC 40}}(x_1), H_{\text{FTSE.MIB}}(x_2)\big) \end{aligned}$$

Where $H_{\text{CAC 40}}(x_1)$ and $H_{\text{FTSE.MIB}}(x_2)$ refer respectively to CAC 40 and FTSE MIB ECDF at their stress loss point; $C\big(H_{\text{CAC 40}}(x_1), H_{\text{FTSE.MIB}}(x_2)\big)$ is copula cumulative distribution function at CAC 40 and FTSE MIB at their stress loss point.[29]

This method is very straightforward and flexible to be applied, allowing to investigate various types of market losses and their likelihood. For example, we could be interested in knowing which is the probability of a 5% CAC 40 decrease while FTSE MIB shows a 3% plunge. The probability of this simultaneous decrease is 0.31%.

Now we analyze a 10% simultaneous stress move:

STOCK INDEX	PRICE	PORTFOLIO VALUE	STRESS MOVE	STRESS LOSS	PROBABILITY
CAC 40	7420	100 000	10%	10 000	0.018%
FTSE.MIB	30440	100 000	10%	10 000	

Table 19 Stress test (stress loss equal to 10%)

In this scenario, we observe a relevant intensification in large losses experienced by our portfolio. For instance, we expect a 20 000€ loss with 0.018% probability. Likelihood of

[29] The use of copulas in risk management (Stander) p.43

experiencing such a loss is notably decreased if compared to 5% downturn, underscoring rarity and magnitude of this particular event.

Lastly, we analyze VaR_{α} and ES_{α} with α= 0.95 and 0.99. We already introduced these kinds of risk measures in previous chapter, now we are going to analyze their risk management application and how they can be useful to state frequency and magnitude extreme movement in market prices.

CAC 40 (α)	$\mathbf{VaR}_{\boldsymbol{\alpha}}$	$\boldsymbol{ES}_{\boldsymbol{\alpha}}$
0.95	1.94%	2.97%
0.99	3.51%	4.91%

FTSE MIB (α)	$\mathbf{VaR}_{\boldsymbol{\alpha}}$	$\boldsymbol{ES}_{\boldsymbol{\alpha}}$
0.95	2.24%	3.27%
0.99	3.78%	5.22%

Table 20 VaR_{α} and ES_{α} for CAC 40 and FTSE MIB

Both risk measures are computed on a daily horizon. In CAC 40 risk scenario, $\mathrm{VaR}_{0.95}$ indicates a loss equal or higher than 1.94% with 5% probability. On the other hand, $ES_{0.95} = 2.97\%$ is the average stock index loss given $\mathrm{VaR}_{0.95}$ is surpassed. Obviously, with α= 0.99 we observe higher intensity in loss magnitude while on the same time we notice a lower probability to happen.
If we compare VaR_{α} and ES_{α} between CAC 40 and FTSE MIB we can notice higher magnitude for the same probability. Once again, FTSE MIB emerges as the riskier stock index between the two. We created a simple system capable of checking various level of losses and their probabilities. What is innovative is its ability to assess with which probability we are going to experience a certain simultaneous loss. For example, we can compute $\mathrm{VaR}_{0.99}$ for each stock index and utilize this amount as possible stress loss. Then, we can determine probability of simultaneous loss in the two stock indexes.

Conclusion

In conclusion, our CAC 40 and FTSE MIB analysis clearly denied normality assumption. During this thesis empirical data show features representing departure from normality. Fat tails and nonlinear dependence characterizing our data need alternative distribution models to capture their complexities. We start by fitting student’s t distribution to CAC 40 and FTSE MIB negative returns suggest that data are characterized by heavy tails, validated by optimal goodness of fit results. Furthermore, extreme values theory applications including POT and BM confirm extreme nature of the data revealed by Student’s t fit.

These alternative representations along with ECDF offer us a new point of view on dependence between CAC 40 and FTSE MIB. In fact, we created various copulas starting by detailed marginal distributions. In this case copula functions show how interdependence evolves when market conditions deteriorate, consistently observing left asymmetric dependence.

Each of this distribution models helped us to compute risk measure that could quantify magnitude and probability of extreme events. For instance, we used classical risk measures such as VaR and ES and less known such as return level and period. All these risk measures contribute to identify potential extreme losses and thus contribute assessing risk management on a portfolio composed by listed companies.

In the last paragraph, we focus on stress testing concept by exploring impact of 5% and 10% simultaneous drop in CAC 40 and FTSE MIB prices. These analyses show two possible adverse scenarios and their probability to happen. Stress tests are crucial for financial firms and institutions in assessing financial stability.

In summary, our study on risk management highlights normality assumption limitations when dealing with financial markets. On the other hand, our thesis unveils cruciality of alternative model distributions such as student’s t and extreme value theory applications. Through risk measures, copulas and stress testing we offer a comprehensive framework to evaluate and manage risks in financial portfolios, thus providing a tool, for market experts, able to mitigate or avoid the impact of market extreme events effectively.

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(https://www.bis.org/fsi/fsisummaries/stress_testing.pdf, s.d.)

Tables and graphs

Thesis Code (R language):

```
library(quantmod)
library(zoo)
library(lubridate)
library(car)
library(copula)
library(tseries)
library(nortest)
myStocks <-lapply(c("^FCHI", "FTSEMIB.MI"), function(x) {getSymbols(x,
from = "2004/12/31","daily", )
FR_data <- myStocks[[1]]
ITA_data <- myStocks[[2]]
head(FR_data)
head(ITA_data)
tail(FR_data)
tail(ITA_data)
length(FR_data)
length(ITA_data)

# Find positions of missing values in France data
missing_positions_FR <- which(is.na(FR_data), arr.ind = TRUE)
# Find positions of missing values in Italy data
missing_positions_ITA <- which(is.na(ITA_data), arr.ind = TRUE)
# Print the positions
print("Missing Positions in France Data:")
print(missing_positions_FR)
print("Missing Positions in Italy Data:")
print(missing_positions_ITA)
which(is.na(ITA_data$FTSEMIB.MI.Adjusted), arr.ind = TRUE)
which(is.na(FR_data$FCHI.Adjusted), arr.ind = TRUE)
to = "2022/12/31",
periodicity = auto.assign=FALSE)}

F_Ret <- c(-diff(FR_data$FCHI.Adjusted)/FR_data$FCHI.Adjusted[-1]*100)
head(F_Ret)
F_Ret_Date <- data.frame(date = as.Date(index(F_Ret)),
F_Ret = F_Ret$FCHI.Adjusted)

head(F_Ret_Date)
F_Ret_Date
I_Ret <- c(-diff(ITA_data$FTSEMIB.MI.Adjusted) / ITA_data$FTSEMIB.MI.Adjusted[-1] * 100)
I_Ret_Date <- data.frame(date = as.Date(index(I_Ret)),
I_Ret = I_Ret)
head(I_Ret_Date)
which(is.na(F_Ret_Date), arr.ind = TRUE)
which(is.na(I_Ret_Date), arr.ind = TRUE)
F_Ret_Date <- na.omit(F_Ret_Date)
I_Ret_Date <- na.omit(I_Ret_Date)
head(I_Ret_Date)
length(F_Ret_Date$date)
length(I_Ret_Date$date)
plot(I_Ret_Date$FTSEMIB.MI.Adjusted)
#grafico#
plot(I_Ret_Date$date,I_Ret_Date$FTSEMIB.MI.Adjusted, type="l", main="FTSE.MIB Returns",
xlab="t", ylab="r")
plot(FI_RET$date,FI_RET$FITSE.MIB, type="l", main="FTSE.MIB Returns", xlab="t",
ylab="r")
```

```
plot(F_Ret_Date$FCHI.Adjusted)
######missingpositions ret####
# Find positions of missing values in France data
missing_positions_FRret <- which(is.na(F_Ret_Date$FCHI.Adjusted), arr.ind = TRUE)
# Find positions of missing values in Italy data
missing_positions_ITAret <- which(is.na(I_Ret_Date$FTSEMIB.MI.Adjusted), arr.ind = TRUE)
# Print the positions
print("Missing Positions in France Data:")
print(missing_positions_FRret)
print("Missing Positions in Italy Data:")
print(missing_positions_ITAret)
FI_RET <- merge(F_Ret_Date, I_Ret_Date)
head(FI_RET)
length(FI_RET$FCHI.Adjusted)
length(FI_RET$FTSEMIB.MI.Adjusted)
names(FI_RET)[2] <- "CAC 40"
names(FI_RET)[3] <- "FITSE.MIB"
head(FI_RET)
mean(FI_RET$CAC 40)
mean(FI_RET$FITSE.MIB)
print(FI_RET$FITSE.MIB[600:800])
returns_equal_to_minus_100 <- sapply(FI_RET, function(x) any(x == -100))
is_infinite <- sapply(FI_RET, function(x) any(is.infinite(x)))
plot(FI_RET$date,FI_RET$FITSE.MIB, type="l", main="FTSE.MIB Returns", xlab="t",
ylab="r")
geom_hline(yintercept = 20, color = "black", size = 1) +
annotate("text", x = -10, y = 20, label = "cacca", color = "black", size = 4, hjust =
0)
plot(FI_RET$date,FI_RET$CAC 40, type="l", main="CAC 40 Returns", xlab="t",
ylim=c(-10,20),ylab="r")
par(mfrow=c(1,1))
par(mar=c(5, 4, 4, 8) + 0.1)  # Adjust margin for better y-axis label display
#grafico#
plot(FI_RET$FITSE.MIB, type="l", col="blue", main="FTSE.MIB and CAC 40 Returns",
xlab="t", ylab="r", ylim=c(-10,20))
lines(FI_RET$CAC 40, col="red")
legend("topleft", legend=c("FTSE.MIB", "CAC 40"), col=c("blue", "red"), lty=1, cex=0.8)
#summary statistics#
summary(FI_RET$CAC 40)
summary(FI_RET$FITSE.MIB)
sd(FI_RET$CAC 40)
sd(FI_RET$FITSE.MIB)
kurtosis(FI_RET$CAC 40)
kurtosis(FI_RET$FITSE.MIB)
skewness(FI_RET$FITSE.MIB)
skewness(FI_RET$CAC 40)
# Perform Anderson-Darling test
ad.test(FI_RET$CAC 40)
ad.test(FI_RET$FITSE.MIB)
# Perform Kolmogorov-Smirnov test
ks.test(FI_RET$CAC 40, "plnorm")
ks.test(FI_RET$FITSE.MIB, "plnorm")
jarque.bera.test(FI_RET$CAC 40)
jarque.bera.test(FI_RET$FITSE.MIB)
Box.test(FI_RET$CAC 40, lag=1, type = "Ljung-Box")
Box.test(FI_RET$FITSE.MIB, lag=1, type = "Ljung-Box")
#QQ-plot utilizzando qqnorm QUESTO E' QUELLO GIUSTO
qqnorm(FI_RET$CAC 40, main = "NORMAL QQ-PLOT CAC 40", pch = 1, col = "black")
# Aggiungi manualmente la linea normale
qqline(FI_RET$CAC 40, distribution = function(p) qnorm(p, mean = mean(FI_RET$CAC 40), sd =
```

```
sd(FI_RET$CAC 40)), col = "red")
legend('topleft', legend = c('CAC 40 observation','Normal line'), col = c('black',
'red'), lwd = 2)
#grafico#
qqnorm(FI_RET$FITSE.MIB, main = "NORMAL QQ-PLOT FTSE MIB", pch = 1, col = "black")
# Aggiungi manualmente la linea normale
qqline(FI_RET$FITSE.MIB, distribution = function(p) qnorm(p, mean = mean(FI_RET$CAC 40),
sd = sd(FI_RET$CAC 40)), col = "red")
legend('topleft', legend = c('FITSE MIB observation','Normal line'), col = c('black',
'red'), lwd = 2)
library(Dowd)
TQQPlot(FI_RET$CAC 40, df=3)
legend( 'topleft', legend= c('CAC 40 observation', 't-distribution line') ,col =
c('black', 'red'), lwd = 2)
TQQPlot(FI_RET$FITSE.MIB, df = 3.2)
legend( 'topleft', legend= c('FITSE MIB observation', 't-distribution line') ,col =
c('black', 'red'), lwd = 2)
title(main = "QQ PLOT FTSE MIB vs student's t with 4 DoF")
par(mfrow=c(1,1))
#hist cac 40 student and normal#grafico#
hist(FI_RET$CAC 40)
hist(FI_RET$FITSE.MIB)
set.seed(3)
x <- rt(800, df= 3.2)
x2 <- seq(min(x), max(x), length = 50)
fun <- dt(x2, df=3.2)
plot(fun)
hist(FI_RET$CAC 40, prob = TRUE, breaks=100, col = "skyblue3", xlab =
names(FI_RET$CAC 40),
ylim = c(0, max(0.5)), xlim=c(-8,8),
main = paste("Histogram and Student-t curve for CAC 40", names(FI_RET$CAC 40)))
lines(x2, fun, col = 1, lwd = 2)
set.seed(3)
y <- rnorm(800)
y2 <- seq(min(y), max(y), length = 50)
fun2 <- dnorm(y2, mean = mean(y), sd = sd(y))
hist(FI_RET$CAC 40, prob = TRUE, breaks=100, col = "skyblue3", xlab =
names(FI_RET$CAC 40),
ylim = c(0, max(0.5)), xlim=c(-8,8),
main = paste("Histogram and Normal curve for CAC 40", names(FI_RET$CAC 40)))
lines(y2, fun2, col = 1, lwd = 2)
#hist fitse mib#grafico#
hist(FI_RET$FITSE.MIB, prob = TRUE, breaks=200, col = "skyblue3", xlab =
names(FI_RET$FITSE.MIB),
ylim = c(0, 0.4), xlim=c(-8,8),
main = paste("Histogram and Student-t curve for FTSE.MIB",
names(FI_RET$FITSE.MIB)))
lines(x2, fun, col = 1, lwd = 2)
hist(FI_RET$FITSE.MIB, prob = TRUE, breaks=100, col = "skyblue3", xlab =
names(FI_RET$FITSE.MIB),
ylim = c(0, max(fun2)), xlim=c(-8,8),
main = paste("Histogram and Normal curve for FTSE.MIB", names(FI_RET$FITSE.MIB)))
lines(y2, fun2, col = 1, lwd = 2)
bivRet <- merge(FI_RET$CAC 40, FI_RET$FITSE.MIB)
tail(bivRet)
cor.test(FI_RET$CAC 40, FI_RET$FITSE.MIB, method = "pearson", use = "complete.obs")
cor_K <- cor.test(FI_RET$CAC 40, FI_RET$FITSE.MIB, method = "kendall", use =
"complete.obs")
print(cor_K)
cor_spearman <- cor.test(FI_RET$CAC 40, FI_RET$FITSE.MIB, method = "spearman", use =
```

```
"complete.obs")
cor_spearman
#GPD+COPULA#
library(POT)
library(evd)
library(extRemes)
library(quantmod)
library(fExtremes)
library(SpatialExtremes) #for the gev qqplot
library(EnvStats) #for the gev qqplot
library(readxl)
library(lubridate)
library(zoo)
#Plot and correlation checks
plot(FI_RET$CAC 40,FI_RET$FITSE.MIB, pch='.')
abline(lm(FI_RET$CAC 40~FI_RET$FITSE.MIB),col='red',lwd=1)
cor(FI_RET$CAC 40,FI_RET$FITSE.MIB, method='spearman')
#COPULA from two ecdf#
library(VineCopula)
u <- pobs(as.matrix(cbind(-FI_RET$CAC 40,-FI_RET$FITSE.MIB)))[,1]
v <- pobs(as.matrix(cbind(-FI_RET$CAC 40,-FI_RET$FITSE.MIB)))[,2]
selectedCopula <- BiCopSelect(u,v,familyset=NA)
head(selectedCopula)
u1 <- pobs(as.matrix(cbind(-FI_RET$CAC 40, -FI_RET$FITSE.MIB)))[, 1]
u2 <- pobs(as.matrix(cbind(-FI_RET$CAC 40, -FI_RET$FITSE.MIB)))[, 2]
result <- BiCopEstList(u1, u2, familyset = 7)
print(result)
# Set the parameters
par <- selectedCopula$par
par2 <- selectedCopula$par2
taub <-selectedCopula$tau
print(par)
# Specify your copula parameters
par <- 0.81
par2 <- 2.23
# Create a BB1 copula
bb1_copula <- BiCop(family = 7, par = par, par2 = par2)
bb1_copula

#first plot#

simulated_CAC 40 <- quantile(-FI_RET$CAC 40, probs = u)
simulated_FITSE <- quantile(-FI_RET$FITSE.MIB, probs = v)
# Plot the results
plot(-FI_RET$CAC 40, -FI_RET$FITSE.MIB, main='Returns')
points(simulated_CAC 40, simulated_FITSE, col='red')
legend('bottomright',c('Observed','Simulated'),col=c('black','red'),pch=21)
# Estimate marginals parameters
CAC 40_mu <- mean(-FI_RET$CAC 40)
CAC 40_sd <- sd(-FI_RET$CAC 40)
FITSE_mu <- mean(-FI_RET$FITSE.MIB)
FITSE_sd <- sd(-FI_RET$FITSE.MIB)
#downoading data
CAC 40_mu
FITSE_mu
#fitting t student#
fitCAC <- fitdist(distribution = 'std', FI_RET$CAC 40)
fitCAC
fitFtse <- fitdist(distribution = 'std', FI_RET$FITSE.MIB)
```

```
fitFtse
par(mfrow=c(1,1))
#threshold POT#
mrlplot(-FI_RET$CAC 40, u.range = c(-1, 2.5), col = c("green", "black", "green"), nt =
200,main='Cac 40 Mean-excess plot')
warnings
mrlplot(-FI_RET$FITSE.MIB, u.range = c(-1, 2.5), col = c("green", "black", "green"), nt
= 200, main='Ftse Mean-excess plot')
threshCACmrl= 2
threshFtsemrl= 2.5
threshCACq= (quantile(-FI_RET$CAC 40, probs=0.9)+quantile(-FI_RET$CAC 40, probs=0.85))/2
threshFtseq= (quantile(-FI_RET$FITSE.MIB, probs=0.9)+quantile(-FI_RET$FITSE.MIB,
probs=0.85))/2
threshFtseq
#qqplot pareto#
qqparetoPlot(-FI_RET$CAC 40, xi = 0.2836, threshold = threshCACmrl)
qqparetoPlot(-FI_RET$CAC 40, xi = 0.2836, threshold = threshCACq)
qqparetoPlot(-FI_RET$FITSE.MIB, xi = 0.2018, threshold = threshFtsemrl)
qqparetoPlot(-FI_RET$FITSE.MIB, xi = 0.2018, threshold = threshFtseq)
#histogram con var e es#
ggplot(data = data.frame(Returns = -GA_Ret$GOOG.Adjusted), aes(x = Returns)) +
geom_histogram(binwidth = 0.02, fill = "lightblue", color = "lightblue") +
geom_density(color = "red", adjust = 0.001) +
geom_vline(xintercept = 2.66, color = "black", size = 1) +
geom_vline(xintercept = 4.25, color = "black", size = 1) +
annotate("text", x = 2.66, y = 20, label = "VaR(0.95)= 2.66", color = "black", size =
4, hjust = 0) +
annotate("text", x = 4.25, y = 15, label = "ES(0.95)= 4.25", color = "black", size =
4, hjust = 0) +
labs(title = "Histogram of GOOG Returns with Fitted GPD Density") +
geom_hline(yintercept = 20, color = "black", size = 1) +
annotate("text", x = -10, y = 20, label = "cacca", color = "black", size = 4, hjust =

0)

k=length(-FI_RET$CAC 40[-FI_RET$CAC 40 > threshCACmrl])
k
length(-FI_RET$CAC 40)
l=length(-FI_RET$FITSE.MIB[-FI_RET$FITSE.MIB > threshFtsemrl])
l
length(-FI_RET$FITSE.MIB)
#optimal POT fit for CAC 40#
CACfit = gpdFit(as.numeric(-FI_RET$CAC 40), u = 1.5)
summary(CACfit)
sefxi= 0.06803024/67.56
sefbeta= 0.06829971/67.56
plot(CACfit)
print(CACfit)
length((FI_RET$date))
xcacfit= gpdSim(model = list(xi=0.2836882, mu=2, beta= 0.7380928), n=4500)
plot(xcacfit)
length(xcacfit)
class(xcacfit)
class(CACfit)
attributes(xcacfit)
attributes(CACfit)
str(CACfit)
#risk measures:VaR and ES#
gpdSfallPlot(xcacfit,0.99,0.95, 50)
```

```
gpdSfallPlot(CACfit,0.99,0.95, 50)
gpdSfallPlot(CACfit@fit$par.ests, 0.99, 0.95, 50)
gpdSfallPlot(CACfit@fit$par.ests, CACfit@fit$threshold, 0.99, 0.95, 50)
args(gpdSfallPlot)
gpdSfallPlot(CACfit@fit$par.ests, CACfit@fit$threshold, 0.99, 0.95, like.num = 50)
FTSEfit
FTSEfit= gpdFit(as.numeric(-FI_RET$FITSE.MIB), u= 2)
summary(FTSEfit)
plot(FTSEfit)
plot(FTSEfit)
#VAR and ES for pot#
gpdRiskMeasures(FTSEfit)
gpdRiskMeasures(CACfit)
secacxi= 0.05503565/67.56
secacbeta= 0.05521823/67.56
tailPlot(CACfit)
tailPlot(FTSEfit)
threshFtsemrl
gpdRiskMeasures(FTSEfit, prob=0.99)
gpdRiskMeasures(CACfit, prob=0.99)
thresholds <- quantile(-FI_RET$CAC 40, probs = c(0.9, 0.95, 0.99))
thresholds
###xicac=0.283688  betacac=0.73809 ftsexi=0.20175 betaftse=0.88126 uguali
library(ggplot2)
#HIST X VAR ED ES
ggplot(data = data.frame(Returns = -FI_RET$CAC 40), aes(x = Returns)) +
geom_histogram(binwidth = 0.02, fill = "lightblue", color = "lightblue") +
geom_density(color = "red", adjust = 0.001) +
geom_vline(xintercept = 2.66, color = "black", size = 1) +
geom_vline(xintercept = 4.25, color = "black", size = 1) +
annotate("text", x = 3, y = 20, label = "VaR(0.95)= 1.94%", color = "black", size = 4,
hjust = 0) +
annotate("text", x = 4.25, y = 15, label = "ES(0.95)= 2.97%", color = "black", size =
4, hjust = 0) +
labs(title = "Histogram of CAC 40 Returns with Fitted GPD Density")
par(mfrow=c(1,1))
#grafico#
ggplot(data = data.frame(Returns = -FI_RET$FITSE.MIB), aes(x = Returns)) +
geom_histogram(binwidth = 0.02, fill = "lightblue", color = "lightblue") +
geom_density(color = "red", adjust = 0.001) +
geom_vline(xintercept = 2.66, color = "black", size = 1) +
geom_vline(xintercept = 4.25, color = "black", size = 1) +
annotate("text", x = 2.66, y = 20, label = " VaR(0.95)= 2.24%", color = "black", size
= 4, hjust = 0) +
annotate("text", x = 4.25, y = 15, label = "ES(0.95)= 3.27%", color = "black", size =
4, hjust = 0) +
labs(title = "Histogram of FTSE MIB Returns with Fitted GPD Density")
# Plot QQ-plots and MRL-plots for both GOOG and AAPL
qqparetoPlot(-FI_RET$CAC 40, xi = 0.2836, threshold = threshCACmrl)
mrlplot(-FI_RET$CAC 40, u.range = c(-1, 2.5), col = c("green", "black", "green"), nt =
200)
qqparetoPlot(-FI_RET$FITSE.MIB, xi = 0.2, threshold = threshFtsemrl)
mrlplot(-FI_RET$FITSE.MIB, u.range = c(-1, 2.5), col = c("green", "black", "green"), nt
= 200)
#Var and ES#
n=4500
k=401
l=304
alpha=0.95
length(FI_RET$CAC 40)
```

```
xiFTSE= 0.2036028
betaFTSE=0.7742930
vriskFTSE= threshFtsemrl + betaFTSE * ( (n*(1-alpha)/l)**(-xiFTSE)-1 ) / xiFTSE
vriskFTSE
esriskFTSE= (vriskFTSE + betaFTSE - xiFTSE*threshFtsemrl)/(1-xiFTSE)
esriskFTSE
threshFtsemrl=2
threshCACmrl=1.5
xiCAC= 0.1901761
betaCAC= 0.7491697
vriskCAC= threshCACmrl + ((n*(1-alpha)/k)**(-xiCAC) -1) * betaCAC/xiCAC
vriskCAC
eriskCAC= (vriskCAC + betaCAC - xiCAC*threshCACmrl)/(1-xiCAC)
eriskCAC
betaCACPOT=0.7380928
xiCACPOT=0.2836882
betaFTSEPOT=0.8812683
xiFTSEPOT=0.2017511
meanPOTCest= 1.795167228
meanPOTFest= 2.01078314
CACQUANTPOT <- qgpd(0.99, loc=threshCACmrl, scale=betaCACPOT, shape=xiCACPOT)
CACQUANTPOT
FTSEQUANTPOT <- qgpd(0.99, loc=threshFtsemrl, scale=betaFTSEPOT, shape=xiFTSEPOT)
FTSEQUANTPOT
portfolioPOT <- data.frame(
Stock = c("CAC 40", "FTSE.MIB"),
Price = c( 7000, 20000),  # Update with actual prices if needed
Nominal = c(-2000, -5000),
StressMove = c(CACQUANTPOT, FTSEQUANTPOT)
)
portfolioPOT
portfolioPOT$StressLoss <- portfolio$Nominal * portfolio$StressMove / 100
portfolioPOT
cdfBB1 <- BiCopCDF(0.99,0.99, family = 7, par = par, par2 = par2)
CAC_BB1Q <- pgpd(9.006515, loc=threshCACmrl, scale=betaCACPOT, shape=xiCACPOT)
FTSE_BB1Q <- pgpd(9.192905, loc=threshFtsemrl, scale=betaFTSEPOT, shape=xiFTSEPOT)
p_stress_movesPOT <- 1+ cdfBB1 - CAC_BB1Q  - FTSE_BB1Q
p_stress_movesPOT
cat("Probability of joint stress moves:", p_stress_movesPOT*100 ,"%" ,"\n")
#stressmoves#
library(creditmodel)
#esempio ch.19#
library(ismev)
# Load necessary libraries
library(copula)
# Calculate daily returns for AAPL and GOOG
CAC_qReturns <- -FI_RET$CAC 40
FTSE_qReturns <-  -FI_RET$FITSE.MIB
tail(CAC_qReturns)
length_C <- length(FI_RET$CAC 40)
length_C
C_BM <- blockMaxima(-FI_RET$CAC 40, block= 22)
C_BM
F_BM <- blockMaxima(-FI_RET$FITSE.MIB, block= 22)
F_BM

C_BM F_BM

gev.fitse
```

```
VaR(F_BM, 0.01,tail="upper")
VaR(gev.fitse, 0.99)
CVaR(C_BM, 0.95)
VaR(F_BM, 0.99)
CVaR(F_BM, 0.95)
#BM plot #
cmax= ts(C_BM, frequency=12, start=c(2005, 1, 1))
length(cmax)
Xn = sort(cmax)
plot.ts(cmax, main='CAC 40 Maxima')
qqPlot(C_BM, distribution= "gevd", estimate.params=TRUE, main="1 Month Maxima",
xlab="GEV Quantiles", ylab="1 Month CAC 40 Maxima Quantiles")
abline(0, 1, col = 'red')
par(mfrow=c(2,2))
fmax= ts(F_BM, frequency=12, start=c(2005, 1, 1))
length(fmax)
Yn = sort(fmax)
plot.ts(fmax, main='FTSE MIB Maxima')
qqPlot(F_BM, distribution= "gevd", estimate.params=TRUE, main="1 Month Maxima",
xlab="GEV Quantiles", ylab="1 Month FITSE MIB Maxima Quantiles")
abline(0, 1, col = 'red')
#altro metodo ma va bene uguale#
par(mfcol = c(4, 1))
gev.cac 40= gevFit(as.numeric(-FI_RET$CAC 40), block = 22, type = "mle")
summary(gev.cac 40)
length(gev.cac 40)
#supergiusti
xigev.cac 40= 0.2973228
mugev.cac 40= 1.6987367
betagev.cac 40= 0.7299249
s.e.xi= 0.06030624/14.40
s.e.mu= 0.05756353/14.40
s.e.beta= 0.04840520/14.40
# probability to observe a loss in the next block larger than maxloss#
MAXloss= max(-FI_RET$CAC 40)
maxloss=10.05267
pExceedcac 40= pgev(4.43, 1.795167228, 1.090684418,  -0.008133936, lower.tail = FALSE)
pExceedcac 40
gev.fitse= gevFit(as.numeric(-FI_RET$FITSE.MIB), block = 22, type = "mle")
summary(gev.fitse)
#giusti, non è vero
xigev.cac 40= -0.008133936
mugev.cac 40= 1.795167228
betagev.cac 40= 1.090684418
xigev.fitse= -0.02224511
mugev.fitse= 2.01078314
betagev.fitse= 1.15902679
#ts giusta negativa
xigev.cac 40= 0.2973228
mugev.cac 40= 1.6987367
betagev.cac 40= 0.7299249
xigev.fitse= 0.216673
mugev.fitse= 1.917174
betagev.fitse= 0.875330
s.e.xif= 0.06084782/14.40
s.e.muf=  0.06966065/14.40
s.e.betaf= 0.05593616/14.40
#BM Risk measures#
MAXloss= max(-FI_RET$FITSE.MIB)
maxlossfitse=10.30386
```

```
pExceedfitse= pgev(4.77, 2.01078314, 1.15902679,  -0.02224511, lower.tail = FALSE)
pExceedfitse

plot(C_BM)

#level which is exceeded in one out of every k blocks of size n#
gevrlevelPlot(gev.cac 40, kBlocks = 12, ci = c(0.95, .9), plottype = c("plot", "add"),
labels = TRUE)
gevrlevelPlot(gev.fitse, kBlocks = 12, ci = c(0.95, .9), plottype = c("plot", "add"),
labels = TRUE)
# Fit the Generalized Extreme Value (GEV) distribution to the max returns
u=5   #set the loss
kreturncac= 1/pgev(4.43, 1.795167228, 1.090684418,  -0.008133936, lower.tail = FALSE)
kreturncac
u=5   #set the loss
kreturnfitse= 1/pgev(4.77, 2.01078314, 1.15902679,  -0.02224511, lower.tail = FALSE)
kreturnfitse
CAC_GEVDist <- gevFit(C_BM)
FTSE_GEVDist <- gevFit(F_BM)
CAC_GEVDist
FTSE_GEVDist
summary(CAC_GEVDist)
#stress test using BM#
meanBMC <- mean(C_BM)
sdBMC <- sd(C_BM)
meanBMC
sdBMC
meanBMF <- mean(F_BM)
sdBMF <- sd(F_BM)
meanBMF
sdBMF
betaCACBM=0.7299249
xiCACBM=-0.2973228
betaFTSEBM=0.875330
xiFTSEBM=-0.216673
meanBMCest= 2.409749
meanBMFest=  2.650639
CACQUANT <- qgev(p=0.99, loc=meanBMCest , scale=betaCACBM, shape = xiCACBM)
CACQUANT
CACQUANT24 <- qgev(p=0.99, 2.409749 , 0.7299249, 0.2973228)
CACQUANT24 <- 4.3176
FTSEQUANT <- qgev(p=0.99, loc=meanBMFest , scale=betaFTSEBM, shape = xiFTSEBM)
FTSEQUANT24 <- 4.7343
portfolio <- data.frame(
Stock = c("CAC 40", "FTSE.MIB"),
Price = c( 7420, 30440),  # Update with actual prices if needed
Nominal = c(-10000, -10000),
StressMove = c(CACQUANT24, FTSEQUANT24)
)
portfolio
portfolio$StressLoss <- portfolio$Nominal * portfolio$StressMove / 100
portfolio
#copula x cac fitse= BM x prob
g <- rCopula(200, tawnt2copfi)
cdftawn2 <- pCopula(c(0.99,0.99),tawnt2copfi)
CAC_GCOPQ <- pgev(4.3176, loc=1.6987367, scale=0.7299249, shape=0.2973228)
CAC_GCOPQ <- pgev(6.719771, loc=1.795167228, scale=1.090684418, shape=-0.008133936)
FTSE_GCOPQ <- pgev(4.7343,loc=2.01078314,scale=1.15902679 ,shape=-0.02224511)
##questi giusti##
```

```
CAC_GCOPQ <- pgev(6.912062, 1.6987367,0.7299249,0.2973228)
FTSE_GCOPQ <- pgev(7.270003 ,1.917174,0.875330 ,0.216673)
#fine risk measures bm
#BM copula#
g <- rCopula(200, tawnt2copfi)
cdftawn2 <- pCopula(c(0.9785406,0.9798418),tawnt2copfi)
p_stress_movesBM <- 1+ cdftawn2 - CAC_GCOPQ -FTSE_GCOPQ
p_stress_movesBM <- 1+ cdftawn2 - 0.99-0.99
p_stress_movesBM
cat("Probability of joint stress moves:", p_stress_movesBM*100 ,"%" ,"\n")
portfolio$probOfJointStressMoves <- p_stress_movesBM
portfolio
portfolio <- data.frame(
Stock = c("CAC 40", "FTSE.MIB"),
Price = c( 7420, 30440),  # Update with actual prices if needed
Nominal = c(-10000, -10000),
StressMove = c(8.883084, 8.822931)
)
portfolio
portfolio$StressLoss <- portfolio$Nominal * portfolio$StressMove / 100
portfolio
##STRESS MOVES ftse cac 40 BB1##giusta#
CAC_GCOPQ2 <- quantile(FI_RET$CAC 40, 0.99)
CAC_GCOPQ2 <- 0.9982521
FTSE_GCOPQ2 <- 0.9989076
ecdf_CAC 40 <- ecdf(-FI_RET$CAC 40)
ecdf_FITSE <- ecdf(-FI_RET$FITSE.MIB)
CAC_GCOPQ2 <-ecdf_CAC 40(5)
FTSE_GCOPQ2 <-ecdf_FITSE(3)
#ECDF marginals copula#
BB1copfi <- VC2copula::BB1Copula(param = c(0.81, 2.24))
v <- rCopula(4500, BB1copfi)
cdfBB1 <- pCopula(c(CAC_GCOPQ2,FTSE_GCOPQ2),BB1copfi)
p_stress_movesfi <- 1+ cdfBB1 - CAC_GCOPQ2 -FTSE_GCOPQ2
p_stress_movesfi
dCopula(c(0.1,0.1),BB1copfi)
#copula by loss
u <- pobs(as.matrix(cbind(-FI_RET$CAC 40,-FI_RET$FITSE.MIB)))[,1]
v <- pobs(as.matrix(cbind(-FI_RET$CAC 40,-FI_RET$FITSE.MIB)))[,2]
selectedCopula <- BiCopSelect(u,v,familyset=NA)
head(selectedCopula)
u1 <- pobs(as.matrix(cbind(-FI_RET$CAC 40, -FI_RET$FITSE.MIB)))[, 1]
u2 <- pobs(as.matrix(cbind(-FI_RET$CAC 40, -FI_RET$FITSE.MIB)))[, 2]
result <- BiCopEstList(u1, u2, familyset = 7)
result
#t stud (par=0.88, par2=4.47, tau=0.68)
#copula with BM marginals
C_BM <- blockMaxima(-FI_RET$CAC 40, block= 22)
length(C_BM)
F_BM <- blockMaxima(-FI_RET$FITSE.MIB, block= 22)
length(F_BM)
uBM <- pobs(as.matrix(cbind(C_BM)))[,1]
vBM <- pobs(as.matrix(cbind(F_BM)))[,1]
stocks_copula1 <-  BiCopSelect(uBM,vBM,familyset=NA)
stocks_copula1
tawnresult <- BiCopEstList(uBM, vBM, familyset =204)
#tawn type 2 (par = 3.21, par2= 0.78, tau = 0.6)#
# Load the copula package
#copula from 2 student's t marginals#
set.seed(123)
```

```
# Simulate data from the provided distributions
marginals <- c("t", "t")
params <- list(list(mean = -0.05515526, sd = 1.47670674, df = 3.00482283),
list(mean = -0.05302316, sd = 1.64009850, df = 3.20365077))
# Simulate data for each marginal distribution
var_a <- pobs(rt(1000, df = params[[1]]$df, ncp = params[[1]]$mean))
var_b <- pobs(rt(1000, df = params[[2]]$df, ncp = params[[2]]$mean))
simulated_data <- cbind(var_a, var_b)
# Fit copula to the data
selectedCopulatawn <- BiCopSelect(simulated_data[, 1], simulated_data[, 2], familyset =
NA)
selectedCopulatawn
# Fit bivariate copula models and compare them
comparison_result <- BiCopEstList(simulated_data[, 1], simulated_data[, 2], familyset =
124)
# Display the summary of log-likelihoods, AICs, and BICs
print(comparison_result)
summary(comparison_result)
#Rotated Tawn type 2 180 degrees (par = 2.02, par2 = 0.01, tau = 0.01) # ll(2.15) aic
(-0.31)  bic(9.51)
# Fit GPD model for CAC 40
CACfit <- gpdFit(as.numeric(-FI_RET$CAC 40), u = 1.5)
summary(CACfit)
simulated_data[, 1] <- qgpd(runif(1000), xi = 0.1901761, beta = 0.7491697)
# Simulate data from GPD distributions
simulated_data31 <- matrix(0, nrow = 1000, ncol = 2)
simulated_data31[, 1] <- rgpd(1000, xi = 0.1901761)
simulated_data31[, 2] <- rgpd(1000, xi = 0.2973228)
# Transform data to copula space (probability space)
u31 <- pobs(simulated_data31)
# Fit copula to the data
selectedCopula <- BiCopSelect(u31[, 1], u31[, 2], familyset = NA)
# Display the selected copula
selectedCopula
comparison_result <- BiCopEstList(u1, u2, familyset = 5 )
print(comparison_result)
#Frank (par = -0.38, tau = -0.04) #  family logLik  2 aic -1.99. bic 2.91
# Simulate data from GPD distributions
library(VineCopula)
# Le tue distribuzioni marginali e i relativi parametri
marginals <- c("t", "t")
params <- list(list(mean = -0.05515526, sd = 1.47670674, df = 3.00482283),
list(mean = -0.05302316, sd = 1.64009850, df = 3.20365077))
# Definisci il modello copula
copula_model <- mvdc(margins = marginals, paramMargins = params)
# Genera dati sintetici
set.seed(123)
obs <- rMvdc(1000, copula_model)
# Visualizza i dati sintetici
print(obs)
#copula from ecdf#
u1 <- pobs(as.matrix(cbind(-FI_RET$CAC 40, -FI_RET$FITSE.MIB)))[, 1]
u2 <- pobs(as.matrix(cbind(-FI_RET$CAC 40, -FI_RET$FITSE.MIB)))[, 2]
result <- BiCopEstList(u1, u2, familyset = 2)
library(gumbel)
alphag= 2.51
u4 <- c(0.99,0.99)
cdfGumbel <- pgumbel(0.99,0.99, 3965,alpha=alphag)
cdfGumbel
CAC_GCOPQ <- pgev(6.719771, loc=1.795167228, scale=1.090684418, shape=-0.008133936)
```

```
FTSE_GCOPQ <- pgev(7.078753,loc=2.01078314,scale=1.15902679 ,shape=-0.02224511)
p_stress_movesBM <- 1+ cdfGumbel - CAC_GCOPQ -FTSE_GCOPQ
p_stress_movesBM
cat("Probability of joint stress moves:", p_stress_movesBM*100 ,"%" ,"\n")
portfolio$probOfJointStressMoves <- p_stress_movesBM

portfolio

par(mfrow=c(1,2))
# Load the copula package
library(copula)
# Define the parameters for the normal copula
coef_ <- 0.8
mycopula <- normalCopula(coef_, dim = 2)
# Define the copulas
frank <- frankCopula(dim=2, param=-0.22)
clayton <- claytonCopula(dim = 2, param = 3.2)
gumbel <- gumbelCopula(dim = 2, param = 10)
clayton
# Define the parameters for the Student's t copula
rho1 <- 0.8
df1 <- 5
stu1 <- tCopula(dim = 2, rho1, df = df1)
persp(tCopula(dim=2,rho1,df=df1),dCopula,xlim = c(0, 1), ylim = c(0, 1), main =
"Student's t Copula PDF")
persp(tCopula(dim=2,rho1,df=df1),pCopula, xlim = c(0, 1), ylim = c(0, 1), main =
"Student's t Copula CDF")
contour(tCopula(dim=2,rho1,df=df1), dCopula, xlim = c(0, 1), ylim = c(0, 1), main =
"Contour Student's t Copula PDF")
contour(tCopula(dim=2,rho1,df=df1), pCopula, xlim = c(0, 1), ylim = c(0, 1), main =
"Contour Student's t Copula CDF")
# Construct multivariate distributions using mvdc
mv_frank <- mvdc(frank, c("norm", "norm"),
paramMargins = list(list(mean = 3, sd = 3), list(mean = 4, sd = 1)))
mv_clayton <- mvdc(clayton, c("norm", "norm"),
paramMargins = list(list(mean = 0, sd = 1), list(mean = 0, sd = 1)))
mv_gumbel <- mvdc(gumbel, c("norm", "norm"),
paramMargins = list(list(mean = 3, sd = 2), list(mean = 2, sd = 3)))
mv_stu <- mvdc(stu1, c("norm", "norm"),
paramMargins = list(list(mean = 0, sd = 1.3), list(mean = 0, sd = 1.2)))
# Create 3D CDF plots for the multivariate distributions
persp(mv_frank, pMvdc, xlim = c(0, 1), ylim = c(0, 1), main = "Frank Copula CDF")
persp(mv_clayton, pMvdc, xlim = c(0, 1), ylim = c(0, 1), main = "Clayton Copula CDF")
persp(mv_gumbel, pMvdc, xlim = c(0, 1), ylim = c(0, 1), main = "Gumbel Copula CDF")
persp(mv_stu, pMvdc, xlim = c(0, 1), ylim = c(0, 1), main = "Student's t Copula CDF")
# Create 3D PDF plots for the multivariate distributions
persp(mv_frank, dMvdc, xlim = c(0, 1), ylim = c(0, 1), main = "Frank Copula PDF")
persp(mv_clayton, dMvdc, xlim = c(0, 1), ylim = c(0, 1), main = "Clayton Copula PDF")
persp(mv_gumbel, dMvdc, xlim = c(0, 1), ylim = c(0, 1), main = "Gumbel Copula PDF")
persp(mv_stu, dMvdc, xlim = c(0, 1), ylim = c(0, 1), main = "Student's t Copula PDF")
contour(frankCopula(0.8,2), dCopula)
persp(frankCopula(3,2),  dCopula, zlim=c(0,2.5))
# Create contour plots for PDF and CDF
contour(mv_frank, dMvdc, xlim = c(0, 1), ylim = c(0, 1), main = "Contour Frank Copula
PDF")
contour(mv_frank, pMvdc, xlim = c(0, 1), ylim = c(0, 1), main = "Contour Frank Copula
CDF")
contour(mv_clayton, dMvdc, xlim = c(0, 1), ylim = c(0, 1), main = "Contour Clayton
Copula PDF")
```

```
contour(mv_clayton, pMvdc, xlim = c(0, 1), ylim = c(0, 1), main = "Contour Clayton
Copula CDF")
contour(mv_gumbel, dMvdc, xlim = c(0, 1), ylim = c(0, 1), main = "Contour Gumbel Copula
PDF")
contour(mv_gumbel, pMvdc, xlim = c(0, 1), ylim = c(0, 1), main = "Contour Gumbel Copula
CDF")
contour(mv_stu, dMvdc, xlim = c(0, 1), ylim = c(0, 1), main = "Contour Student's t
Copula PDF")
contour(mv_stu, pMvdc, xlim = c(0, 1), ylim = c(0, 1), main = "Contour Student's t
Copula CDF")
#pag198#
rhol=0.3
dfl=4
stulibro <- tCopula(dim = 2, rhol, df = dfl)
mv_stulibro <- mvdc(stu1, c("norm", "norm"),
paramMargins = list(list(mean = 0, sd = 1), list(mean = 0, sd = 1)))
persp(mv_stulibro, dMvdc, xlim = c(0, 1), ylim = c(0, 1), main = "Student's t Copula
libro PDF")
persp(mv_stulibro, pMvdc, xlim = c(0, 1), ylim = c(0, 1), main = "Student's t Copula
libro PDF")
#Bivariate copula: BB1 (par = 0.54, par2 = 1.29, tau = 0.39)
#Bivariate copula: t (par = 0.56, par2 = 3.06, tau = 0.38)
#Bivariate copula: Gumbel (par = 1.57, tau = 0.36)
persp(gumbelCopula(1.57,2), dCopula, main = "APPLE/GOOGLE Copula PDF (Gumbel)")
contour(gumbelCopula(1.57,2), dCopula, main = "APPLE/GOOGLE Copula PDF (Gumbel)")
persp(tCopula(dim=2,0.8,df=3), dCopula, main = "APPLE/GOOGLE Copula (t)")
contour(tCopula(dim=2,0.8,df=3), pCopula, main = "APPLE/GOOGLE Copula (t)")
persp(claytonCopula(1,2),dCopula)
contour
#NO MARGINS PREIMPOSTATI#

#normal#

persp(normalCopula(0.8,2), dCopula, main = "Normal Copula PDF")
contour(normalCopula(0.2,2), dCopula,main = "Normal Contour Copula PDF")
persp(normalCopula(0.2,2), pCopula, main = "Normal Copula CDF")
contour(normalCopula(0.5,2), pCopula,main = "Normal Contour Copula CDF")
#studentt#
persp(tCopula(dim=2,0.56,df=3),dCopula,xlim = c(0, 1), ylim = c(0, 1), main = "Student's
t Copula PDF")
contour(tCopula(dim=2,0.8,df=3), dCopula, xlim = c(0, 1), ylim = c(0, 1), main =
"Contour Student's t Copula PDF")
persp(tCopula(dim=2,0.8,df=3),pCopula, xlim = c(0, 1), ylim = c(0, 1), main = "Student's
t Copula CDF")
contour(tCopula(dim=2,0.8,df=3), pCopula, xlim = c(0, 1), ylim = c(0, 1), main =
"Contour Student's t Copula CDF")
contour(tCopula(dim=2,0.8,df=14),pCopula, xlim = c(0, 1), ylim = c(0, 1), main =
"Contour Student's t Copula PDF")
#clayton#
persp(claytonCopula(0.8,2), dCopula, main = "Clayton Copula PDF")
contour(claytonCopula(0.8,2), dCopula,main = "Contour Clayton Copula PDF")
persp(claytonCopula(0.8,2), pCopula, main = "Clayton Copula CDF")
contour(claytonCopula(0.8,2), pCopula,main = "Contour Clayton Copula CDF")
#gumbel# non so come mai ma va solo con numeri interi#
gumby=0.8
dimy=2
persp(gumbelCopula(8,2), dCopula, main = "Gumbel Copula PDF")
contour(gumbelCopula(2,2), dCopula,main = "Contour Gumbel Copula PDF")
persp(gumbelCopula(8,2), pCopula, main = "Gumbel Copula CDF")
```

```
contour(gumbelCopula(2,2), pCopula,main = "Contour Gumbel Copula CDF")
#frank copula cacfitse#
persp(frankCopula(-0.38,2), dCopula, main = "Frank Copula PDF")
contour(frankCopula(-0.38,2), dCopula, main = "Contour Frank Copula PDF")
persp(frankCopula(-0.38,2), pCopula,main = "Contour Frank Copula CDF")
contour(frankCopula(-0.38,2), pCopula, main = "Contour Frank Copula CDF")
#BB1 copula cac 40 fitse#
bb1vc2studentcac <- VC2copula::BB1Copula(param = c(0.81, 2.24))
mv_bb1vc2studentcac <- mvdc(bb1vc2studentcac, c("t", "t"),
paramMargins = list(list(df = 3), list(df = 3)))
persp(mv_bb1vc2studentcac, dMvdc, xlim = c(0, 1), ylim = c(0, 1), main = "BB1 Copula
PDF")
persp(mv_bb1vc2studentcac, pMvdc, xlim = c(0, 1), ylim = c(0, 1), main = "BB1 Copula
CDF")
contour(mv_bb1vc2studentcac, dMvdc, xlim = c(0, 1), ylim = c(0, 1), main = "BB1 Copula
PDF")
contour(mv_bb1vc2studentcac, pMvdc, xlim = c(0, 1), ylim = c(0, 1), main = "BB1 Copula
CDF")
tawnt2copfi <- VC2copula::tawnT2Copula(param = c(3.21, 0.78))
mv_tawnt2copfi <- mvdc(tawnt2copfi, c("t", "t"),
paramMargins = list(list(df = 3), list(df = 3)))
persp(mv_tawnt2copfi, dMvdc, xlim = c(0, 1), ylim = c(0, 1), main = "Tawn Copula PDF")
persp(mv_tawnt2copfi, pMvdc, xlim = c(0, 1), ylim = c(0, 1), main = "Tawn Copula CDF")
contour(mv_tawnt2copfi, dMvdc, xlim = c(0, 1), ylim = c(0, 1), main = "TawnCopula PDF")
contour(mv_tawnt2copfi, pMvdc, xlim = c(0, 1), ylim = c(0, 1), main = "Tawn Copula CDF")
#tawnt2cop CAC 40FITSE#
# Creating Tawn Type 2 copula object with parameters c(2, 0.5)
copula_obj <- VC2copula::tawnT2Copula(param = c(3.21, 0.78))
# Visualizing the copula
persp(copula_obj, dCopula, zlim = c(0, 14))
copula_objt2 <- VC2copula::r90TawnT1Copula(param = c(-13.36,0))
persp(copula_objt2, dCopula, zlim = c(0, 2) )
persp(copula_objt2, pCopula, zlim = c(0, 1) )
n <- 3965 # Number of observations
#simulated_data12 <- simu.BB1(3965, alpha=1,d=2,scale1=1,scale2=2,shape1=0.5,shape2=2)
simulated_data1 <- BiCop(3965,214,2.02,0.01)
persp(simulated_data1, dMvdc, xlim = c(0, 1), ylim = c(0, 1), main = "BB1 Copula PDF")
# Create a copula object for a Tawn type 2 copula
copula_objectfi <- BiCop(214, 2.02, 0.01)
# Simulate data from the copula using BiCopSim
simulated_data1fi <- BiCopSim(1000, copula_objectfi)
contour(simulated_data1fi, dMvdc, xlim = c(0, 1), ylim = c(0, 1), main = "Tawn Type 2
Copula PDF")
# Create a copula object for a Tawn Type 2 copula
copula_object <- BiCop(214, 2.02, 0.01)
# Simulate data from the copula using BiCopSim
simulated_data <- BiCopSim(4000, copula_object)
# Plot the copula density using persp
persp(simulated_data, dMvdc, xlim = c(0, 1), ylim = c(0, 1), main = "Tawn Type 2 Copula
PDF")
plot(simulated_data, dMvdc)
length(simulated_data)
persp(simulated_data[, 1],simulated_data[, 2], pch = '.', col = 'red')
plot(simulated_data[, 1],simulated_data[, 2], pch = '.', col = 'red')
##secondo pagina 19 di Copula Models for Dependence: Comparing Classical and Bayesian
Approaches
##Lídia Maria Branco Correia Martins Andréè sbagliata solo frank
##questa gumbel viene diversa
mvNN <- mvdc(gumbelCopula(3), c("norm", "norm"),
list(list(mean = 0, sd = 1), list(mean = 1)))
```

```
persp(mvNN, dMvdc, xlim=c(0,1), ylim=c(0,1), zlim=c(0,1),main = "Density")
persp(mvNN, pMvdc, xlim=c(-2, 2), ylim=c(-1, 3), main = "Cumulative Distr.")
contour(mvNN, dMvdc, xlim=c(-2, 2), ylim=c(-1, 3), main = "Density")
contour(mvNN, pMvdc, xlim=c(-2, 2), ylim=c(-1, 3), main = "Cumulative Distr.")
par(mfrow=c(1,2))
```

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